

# Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations with weak dissipation

Marina Ghisi

Università degli Studi di Pisa

Dipartimento di Matematica “Leonida Tonelli”

PISA (Italy)

e-mail: [ghisi@dm.unipi.it](mailto:ghisi@dm.unipi.it)

## Abstract

We consider Kirchhoff equations with a small parameter  $\varepsilon$  such as

$$\varepsilon u_\varepsilon''(t) + (1+t)^{-p} u_\varepsilon'(t) + |A^{1/2} u_\varepsilon(t)|^{2\gamma} A u_\varepsilon(t) = 0.$$

We prove the existence of global solutions when  $\varepsilon$  is small with respect to the size of initial data, for all  $0 \leq p \leq 1$  and  $\gamma \geq 1$ . Then we provide global-in-time error estimates on  $u_\varepsilon - u$  where  $u$  is the solution of the parabolic problem obtained setting formally  $\varepsilon = 0$  in the previous equation.

**Mathematics Subject Classification 2000 (MSC2000):** 35B25, 35L70, 35B40.

**Key words:** hyperbolic-parabolic singular perturbation, Kirchhoff equations, weak dissipation, quasilinear hyperbolic equations, degenerate hyperbolic equations.

# 1 Introduction

Let  $H$  be a real Hilbert space. For every  $x$  and  $y$  in  $H$ ,  $|x|$  denotes the norm of  $x$ , and  $\langle x, y \rangle$  denotes the scalar product of  $x$  and  $y$ . Let  $A$  be a self-adjoint linear operator on  $H$  with dense domain  $D(A)$ . We assume that  $A$  is nonnegative, namely  $\langle Ax, x \rangle \geq 0$  for every  $x \in D(A)$ , so that for every  $\alpha \geq 0$  the power  $A^\alpha x$  is defined provided that  $x$  lies in a suitable domain  $D(A^\alpha)$ .

For every  $\varepsilon > 0$  we consider the Cauchy problem

$$\varepsilon u_\varepsilon''(t) + (1+t)^{-p} u_\varepsilon'(t) + |A^{1/2} u_\varepsilon(t)|^{2\gamma} A u_\varepsilon(t) = 0, \quad (1.1)$$

$$u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1. \quad (1.2)$$

Equation (1.1) is the prototype of all degenerate Kirchhoff equations with weak dissipation, such as

$$\varepsilon u_\varepsilon''(t) + (1+t)^{-p} u_\varepsilon'(t) + m(|A^{1/2} u_\varepsilon(t)|^2) A u_\varepsilon(t) = 0 \quad \forall t \geq 0, \quad (1.3)$$

where  $m : [0, +\infty[ \rightarrow [0, +\infty[$  is a given function which is always assumed to be of class  $C^1$ . It is well known that (1.3) is the abstract setting of a quasilinear nonlocal partial differential equation of hyperbolic type which was proposed as a model for small vibrations of strings and membranes.

Let us start by recalling some general terminology. Equation (1.3) is called *nondegenerate* (or strictly hyperbolic) when

$$\mu := \inf_{\sigma \geq 0} m(\sigma) > 0,$$

and *mildly degenerate* when  $\mu = 0$  but  $m(|A^{1/2} u_0|^2) \neq 0$ . In the special case of equation (1.1) this assumption reduces to

$$A^{1/2} u_0 \neq 0. \quad (1.4)$$

Concerning the dissipation term, we have *constant dissipation* when  $p = 0$  and *weak dissipation* when  $p > 0$ . Finally, the operator  $A$  is called *coercive* when

$$\nu := \inf \left\{ \frac{\langle Ax, x \rangle}{|x|^2} : x \in D(A), x \neq 0 \right\} > 0,$$

and *noncoercive* when  $\nu = 0$ .

In the following we recall briefly what is a singular perturbation problem and the “state of the art”. For a more complete discussion on this argument we refer to the survey [11] and to the references contained therein. Moreover we concentrate mostly on equation (1.1) recalling only a few facts on the general equation (1.3).

The singular perturbation problem in its generality consists in proving the convergence of solutions of (1.1), (1.2) to solutions of the first order problem

$$u'(t) + (1+t)^p |A^{1/2} u(t)|^{2\gamma} A u(t) = 0, \quad u(0) = u_0, \quad (1.5)$$

obtained setting formally  $\varepsilon = 0$  in (1.1), and omitting the second initial condition in (1.2). In the concrete case, equation in (1.5) is a partial differential equation of parabolic type. With a little abuse of notation in the following we refer to hyperbolic and parabolic problems (or behavior) also in the abstract setting of equations (1.1) and (1.5).

Following the approach introduced by J. L. Lions [14] in the linear case, one defines the corrector  $\theta_\varepsilon(t)$  as the solution of the second order *linear* problem

$$\varepsilon \theta_\varepsilon''(t) + (1+t)^{-p} \theta_\varepsilon'(t) = 0 \quad \forall t \geq 0, \quad (1.6)$$

$$\theta_\varepsilon(0) = 0, \quad \theta_\varepsilon'(0) = u_1 + |A^{1/2}u_0|^{2\gamma} Au_0 =: w_0. \quad (1.7)$$

It is easy to see that  $\theta_\varepsilon'(0) = u_\varepsilon'(0) - u'(0)$ , hence this corrector keeps into account the boundary layer due to the loss of one initial condition. Finally one defines  $r_\varepsilon(t)$  and  $\rho_\varepsilon(t)$  in such a way that

$$u_\varepsilon(t) = u(t) + \theta_\varepsilon(t) + r_\varepsilon(t) = u(t) + \rho_\varepsilon(t) \quad \forall t \geq 0. \quad (1.8)$$

With these notations, the singular perturbation problem consists in proving that  $r_\varepsilon(t) \rightarrow 0$  or  $\rho_\varepsilon(t) \rightarrow 0$  in some sense as  $\varepsilon \rightarrow 0^+$ . In particular *time-independent* estimates on  $\rho_\varepsilon(t)$  or  $r_\varepsilon(t)$  as  $\varepsilon \rightarrow 0^+$  are called *global error estimates*.

In this paper we restrict ourself to the so called parabolic regime, namely to the case where  $0 \leq p \leq 1$ . The reason is that equations (1.1) and (1.3) have a different behavior when  $p \leq 1$  or  $p > 1$ . This is true also in the linear nondegenerate case. Let us indeed consider equation

$$au''(t) + \frac{b}{(1+t)^p} u'(t) + cAu(t) = 0, \quad (1.9)$$

where  $a, b, c$  are positive parameters, and  $p \geq 0$ . This equation was investigated by T. Yamazaki [25] and J. Wirth [23]. They proved that (1.9) has both parabolic and hyperbolic features, and which nature prevails depends on  $p$ . When  $p < 1$  the equation has parabolic behavior, in the sense that all its solutions decay to 0 as  $t \rightarrow +\infty$  as solutions of the parabolic equation with  $a = 0$ . When  $p > 1$  the same equation has hyperbolic behavior, meaning that every solution is asymptotic to a suitable solution of the non-dissipative equation with  $b = 0$  (and in particular all non-zero solutions do not decay to zero). In the critical case  $p = 1$  the nature of the problem depends on  $b/a$ , with the parabolic behavior prevailing as soon as the ratio is large enough. In [9] and [10] it was proved that also in the case of Kirchhoff equation we have always hyperbolic behavior when  $p > 1$ , meaning that non-zero global solutions (provided that they exist) cannot decay to 0. On the other hand, solutions of the limit parabolic problem decay to zero also for  $p > 1$ , faster and faster as  $p$  grows.

The study of the singular perturbation problem has generated a considerable literature in particular regarding the preliminary problem of the existence of global solutions for (1.1) (or (1.3)). Despite of this, existence of global solutions without smallness assumptions on  $\varepsilon$  is a widely open question. The existence of global solutions for (1.1),

(1.2) in the case of a constant dissipation ( $p = 0$ ) and  $\gamma \geq 1$  when  $\varepsilon$  is small and (1.4) holds true, was established by K. Nishihara and Y. Yamada [19] (see also E. De Brito [3] and Y. Yamada [24] for the nondegenerate case, [5] for the general case and [4] for the case  $\gamma < 1$ ). Moreover optimal and  $\varepsilon$ -independent decay estimates were obtained in [7] (and by T. Mizumachi ([16, 17]) and K. Ono ([20, 21]) when  $\gamma = 1$ ).

When  $0 \leq p \leq 1$  the existence of global solutions, always for  $\varepsilon$  small, in the non-degenerate case was proved in recent years by M. Nakao and J. Bae [18], by T. Yamazaki [26, 27], and in [9].

The first result for (1.1) when  $p > 0$  was obtained by K. Ono [22]. In the special case  $\gamma = 1$  he proved that a global solution exists provided that  $\varepsilon$  is small and  $p \in [0, 1/3]$ . Then for ten years there were no significant progresses. The reason of the slow progress in this field is hardly surprising. In the weakly dissipative case existence and decay estimates have to be proved in the same time. The better are the decay estimates, the stronger is the existence result. This is due to the competition between the smallness of the dissipation term and the one of the nonlinear term. Both of them decay to zero at infinity and it seems fundamental to understand which of them prevails. Ten years ago decay estimates for degenerate equations were far from being optimal, but for the special case  $\gamma = 1$ . In [7] a new method for obtaining optimal decay estimates was introduced and this allowed a substantial progress. In particular in [10] the following result has been proved.

**Theorem 1 ([10])** Let  $(u_0, u_1) \in D(A) \times D(A^{1/2})$ . If the operator  $A$  is coercive and  $0 \leq p \leq 1$ ,  $\gamma > 0$ , for  $\varepsilon$  small the mildly degenerate problem (1.1), (1.2) has a unique global solution such that

$$\begin{aligned} \frac{C_1}{(1+t)^{(p+1)/\gamma}} &\leq |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{(p+1)/\gamma}} \quad \forall t \geq 0, \\ \frac{C_1}{(1+t)^{(p+1)/\gamma}} &\leq |Au_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{(p+1)/\gamma}} \quad \forall t \geq 0, \\ |u'_\varepsilon(t)|^2 &\leq \frac{C_2}{(1+t)^{2+(p+1)/\gamma}} \quad \forall t \geq 0. \end{aligned}$$

If the operator  $A$  is only non negative,  $\gamma \geq 1$  and

$$0 \leq p \leq \frac{\gamma^2 + 1}{\gamma^2 + 2\gamma - 1}, \tag{1.10}$$

for  $\varepsilon$  small the mildly degenerate problem (1.1), (1.2) has a unique global solution such that

$$\begin{aligned} \frac{C_1}{(1+t)^{(p+1)/\gamma}} &\leq |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{(p+1)/(\gamma+1)}} \quad \forall t \geq 0, \\ |Au_\varepsilon(t)|^2 &\leq \frac{C_2}{(1+t)^{(p+1)/\gamma}} \quad \forall t \geq 0, \end{aligned}$$

$$|u'_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{[2\gamma^2+(1-p)\gamma+p+1]/(\gamma^2+\gamma)}} \quad \forall t \geq 0.$$

In the case of noncoercive operators this result is not optimal because of (1.10). This gap is due to the fact that in this second case the estimate on  $|A^{1/2}u_\varepsilon|^2$  is worse. This problem is in some sense unavoidable. Indeed, also in the case of linear equations, small eigenvalues can make worse the decay of solutions. Despite of this, the first result in this paper fills up the gap. The key technical point is that in the noncoercive case a better decay of  $|Au_\varepsilon|^2$  compensates a worse decay of  $|A^{1/2}u_\varepsilon|^2$  and the two decay rates are strictly related (see Proposition 3.7). This unexpected decay property requires some new and subtle estimates. Such estimates improve the decay rates also when (1.10) is satisfied.

Once we know that a global solution of (1.1) exists, we can focus on the singular perturbation problem. This question was solved in the nondegenerate case. In that case decay - error estimates were proved, that consist in estimating in the same time the behavior of  $u_\varepsilon(t) - u(t)$  as  $t \rightarrow +\infty$  and as  $\varepsilon \rightarrow 0^+$  (see H. Hashimoto and T. Yamazaki [13], T. Yamazaki [26, 27] and [9]). On the contrary the singular perturbation problem is still quite open in the degenerate case. With respect to global in time error estimates we indeed know only the following partial result in the constant dissipative case (see [8] where however more general nonlinearities are considered).

**Theorem 2 (Constant dissipation, [8])** If we assume that  $p = 0$ ,  $\gamma \geq 1$ ,  $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$ , then there exists a constant  $C$  such that for every  $\varepsilon$  small we have that

$$|\rho_\varepsilon(t)|^2 + \varepsilon |A^{1/2}\rho_\varepsilon(t)|^2 \leq C\varepsilon^2 \quad \forall t \geq 0,$$

$$\int_0^{+\infty} |r'_\varepsilon(t)|^2 dt \leq C\varepsilon.$$

This result is far from being optimal. First of all only the convergence rate of  $|\rho_\varepsilon(t)|^2$  is optimal, while with these regularity assumptions on the initial data one can expect an optimal convergence rate also for  $|A^{1/2}\rho_\varepsilon|^2$  (see [6]). Moreover it is limited to equations with constant dissipation. The second result of this paper fills up completely this gap with respect to error estimates in the case of coercive operators or when  $p$  verifies (1.10) (hence always in the case of a constant dissipation) and provides global, but not optimal, error estimates in the remaining cases. Also to prove this second result are fundamental decay estimates as accurate as possible. Conversely it is still open the problem of decay-error estimates in all degenerate cases.

This paper is organized as follows. In Section 2 we state precisely our results. Section 3 is devoted to the proofs, and it is divided into several parts. In particular to begin with in Section 3.1 we state and prove some general lemmata, then in Section 3.2 we consider the parabolic problem (1.5) and finally in Sections 3.3 and 3.4 we prove the results.

## 2 Statements

The first result we state concerns the existence of global solutions for (1.1) and their decay properties.

**Theorem 2.1 (Global solutions and decay)** *Let us assume that  $0 \leq p \leq 1$ ,  $\gamma \geq 1$  and  $A$  be a nonnegative operator. Let us assume that  $(u_0, u_1) \in D(A) \times D(A^{1/2})$  satisfy (1.4).*

*Then there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \in (0, \varepsilon_0)$  problem (1.1), (1.2) has a unique global solution*

$$u_\varepsilon \in C^2([0, +\infty[; H) \cap C^1([0, +\infty[; D(A^{1/2})) \cap C^0([0, +\infty[; D(A)).$$

*Moreover there exist positive constants  $C_1$  and  $C_2$  such that*

$$|u_\varepsilon(t)|^2 \leq C_1 \quad \forall t \geq 0; \quad (2.1)$$

$$\frac{C_1}{(1+t)^{(p+1)/\gamma}} \leq |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{(p+1)/(\gamma+1)}} \quad \forall t \geq 0; \quad (2.2)$$

$$|A^{1/2}u_\varepsilon(t)|^{2(\gamma-1)} |Au_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{(p+1)}} \quad \forall t \geq 0; \quad (2.3)$$

$$|u'_\varepsilon(t)|^2 \leq \frac{C_2 |A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}}{(1+t)^{1-p}} \quad \forall t \geq 0; \quad (2.4)$$

$$\int_0^{+\infty} |u'_\varepsilon(t)|^2 (1+t) dt \leq C_2. \quad (2.5)$$

**Remark 2.2** Inequality (2.3) as far we know is new and it is the core of the existence theorem. This estimate says that in the case when the operator  $A$  is noncoercive it is of course possible that  $|A^{1/2}u_\varepsilon|$  decays slow, but in this case  $|Au_\varepsilon|$  decays stronger than in the coercive case.

**Remark 2.3** When the operator  $A$  is noncoercive,  $\gamma = 1$  is the only case in which this result is contained in Theorem 1 since in the other cases  $(\gamma^2 + 1)/(\gamma^2 + 2\gamma - 1) < 1$  and moreover there is always an improvement on the decay rates (see (2.3) and (2.4)).

The next result regards error estimates. It is divided into two parts. The first one concerns all non negative operators and all  $0 \leq p \leq 1$  and the exponent of  $\varepsilon$  in the estimates is not optimal. The second one gives optimal estimates but with some restrictions on the operator or on the admissible values of  $p$ .

**Theorem 2.4 (Global-in-time error estimates)** *Let us assume that  $0 \leq p \leq 1$ ,  $\gamma \geq 1$  and  $A$  be a nonnegative operator. Let  $u_\varepsilon(t)$  be the solution of equation (1.1) with initial data  $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$  satisfying (1.4). Let  $u(t)$  be the solution of the corresponding first order problem (1.5), and let  $r_\varepsilon(t)$  and  $\rho_\varepsilon(t)$  be defined by (1.8).*

*Then we have the following conclusions.*

(1) There exists a constant  $C_3$  such that for every  $\varepsilon$  small enough we have that

$$|\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + \int_0^t \frac{|r'_\varepsilon(s)|^2}{|A^{1/2}u(s)|^{2\gamma}} \frac{1}{(1+s)^p} ds \leq C_3\varepsilon \quad \forall t \geq 0. \quad (2.6)$$

(2) If in addition we assume that

$$A \text{ is coercive or } 0 \leq p \leq \frac{\gamma^2 + 1}{\gamma^2 + 2\gamma - 1} \quad (2.7)$$

then there exists a constant  $C_4$  such that for every  $\varepsilon$  small enough we have that

$$|\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + \int_0^t \frac{|r'_\varepsilon(s)|^2}{|A^{1/2}u(s)|^{2\gamma}} \frac{1}{(1+s)^p} ds \leq C_4\varepsilon^2 \quad \forall t \geq 0. \quad (2.8)$$

**Remark 2.5** When the initial data are more regular it is of course possible to achieve an estimate on  $A\rho_\varepsilon$  like the ones in (2.6) and (2.8). Moreover in this case one can get also estimates on  $r'_\varepsilon$  exactly as in [8], [9] (see also [11]) We do not give here the precise statements and proofs since they only lengthen the paper without introducing new ideas.

**Remark 2.6** In the integrals in (2.6), (2.8) it appears the coefficient  $|A^{1/2}u|^{-2\gamma}$ . When  $A$  is a coercive operator we can replace this term with  $|A^{1/2}u_\varepsilon|^{-2\gamma}$  or  $(1+t)^{p+1}$ , indeed they all have the same behavior. On the contrary when  $A$  is noncoercive the use of  $|A^{1/2}u|^{-2\gamma}$  seems compulsory.

### 3 Proofs

Proofs are organized as follows. First of all in Section 3.1 we state and prove some general lemmata that do not concern directly the Kirchhoff equation. In Section 3.2 then we recollect all the properties of the solutions of (1.5) we need. Finally in Section 3.3 we prove Theorem 2.1 and in Section 3.4 we prove Theorem 2.4.

#### 3.1 Basic Lemmata

Numerous variants of the following comparison result have already been used in [4, 5, 7, 9, 10] and we refer to these ones for the proof.

**Lemma 3.1** *Let  $T > 0$ , and let  $f : [0, T[ \rightarrow [0, +\infty[$  be a function of class  $C^1$ . Let  $\phi : [0, T[ \rightarrow [0, +\infty[$  be a continuous function. Then the following implications hold true.*

(1) *Let us assume that there exists a constant  $a \geq 0$  such that*

$$f'(t) \leq -\phi(t)\sqrt{f(t)}(\sqrt{f(t)} - a) \quad \forall t \in [0, T[;$$

*then we have that*

$$f(t) \leq \max\{f(0), a^2\} \quad \forall t \in [0, T[.$$



(2) Let us assume that there exists a constant  $a \geq 0$  such that

$$f'(t) \leq -\phi(t)f(t)(f(t) - a) \quad \forall t \in [0, T[;$$

then we have that

$$f(t) \leq \max\{f(0), a\} \quad \forall t \in [0, T[.$$

A proof of the next comparison result is contained in [10] (Lemma 3.2).

**Lemma 3.2** *Let  $w : [0, +\infty[ \rightarrow [0, +\infty[$  be a function of class  $C^1$  with  $w(0) > 0$ . Let  $a > 0$  be a positive constant.*

*Then the following implications hold true.*

(1) *If  $w$  satisfies the differential inequality*

$$w'(t) \leq -a(1+t)^p [w(t)]^{1+\gamma} \quad \forall t \in [0, +\infty[,$$

*then for some constant  $\gamma_1$  we have the following estimate*

$$w(t) \leq \frac{\gamma_1}{(1+t)^{(p+1)/\gamma}} \quad \forall t \in [0, +\infty[.$$

(2) *If  $w$  satisfies the differential inequality*

$$w'(t) \geq -a(1+t)^p [w(t)]^{1+\gamma} \quad \forall t \in [0, +\infty[,$$

*then for some constant  $\gamma_2$  we have the following estimate*

$$w(t) \geq \frac{\gamma_2}{(1+t)^{(p+1)/\gamma}} \quad \forall t \in [0, +\infty[.$$

Let us now state and prove the third lemma.

**Lemma 3.3** *Let  $F, G : [0, T[ \rightarrow [0, +\infty[$  be functions of class  $C^1$ . Let  $\varphi : [0, T[ \rightarrow ]0, +\infty[$  be a continuous function and  $a > 0, b \geq 0, c \geq 0$  be real numbers. Let us assume that in  $[0, T[$  the following inequality holds true:*

$$(F + G)'(t) \leq -\varphi(t)(F(t) + a(G(t))^2 - bG(t) - c(G(t))^{3/2}). \quad (3.1)$$

*Let us set  $\sigma_0 := (c + \sqrt{ab})/a$ . Then we get*

$$G(t) + F(t) \leq \sigma_0^2(1 + b + c\sigma_0) + F(0) + G(0) + 1, \quad \forall t \in [0, T[. \quad (3.2)$$

PROOF. Let us set

$$S := \sup\{t < T : \text{ in } [0, t] \text{ the inequality (3.2) holds true}\}.$$

It is obvious that  $S > 0$ . We want to prove that  $S = T$ . Let us assume by contradiction that  $S < T$ . Therefore in  $[0, S[$  the inequality (3.2) holds true, moreover

$$G(S) + F(S) = \sigma_0^2(1 + b + c\sigma_0) + F(0) + G(0) + 1 \quad (3.3)$$

and

$$(G + F)'(S) \geq 0. \quad (3.4)$$

If  $G(S) > \sigma_0^2$  then

$$a(G(S))^2 - bG(S) - c(G(S))^{3/2} = G(S)(aG(S) - b - \sqrt{G(S)}) > 0. \quad (3.5)$$

Indeed let us set  $y = \sqrt{G(S)}$ , then

$$ay^2 - cy - b > 0 \quad \text{if} \quad y > \frac{c + \sqrt{c^2 + 4ab}}{2a} =: \sigma_1$$

and by assumption  $\sigma_1 \leq \sigma_0 < y$ . Plugging (3.5) in (3.1) we hence arrive at

$$(G + F)'(S) < 0$$

in contrast with (3.4).

Let us now assume that  $G(S) \leq \sigma_0^2$ . Hence from (3.3) we get:

$$\begin{aligned} & F(S) - bG(S) - c(G(S))^{3/2} = \\ & \sigma_0^2(1 + b + c\sigma_0) + F(0) + G(0) + 1 - (b + 1)G(S) - c(G(S))^{3/2} \\ & > \sigma_0^2(1 + b + c\sigma_0) - (b + 1)G(S) - c(G(S))^{3/2} \\ & \geq \sigma_0^2(1 + b + c\sigma_0) - (b + 1)\sigma_0^2 - c\sigma_0^3 = 0. \end{aligned}$$

Hence by (3.1) we obtain once again

$$(G + F)'(S) < 0$$

in contrast with (3.4).  $\square$

The following lemma is essential in the proof of error estimates.

**Lemma 3.4** *Let us assume that  $m : [0, +\infty[ \rightarrow [0, +\infty[$  is a nondecreasing function. Then for all  $x, y \in D(A)$  we get*

$$\langle m(|A^{1/2}x|^2)Ax - m(|A^{1/2}y|^2)Ay, x - y \rangle \geq \frac{1}{2} [m(|A^{1/2}x|^2) + m(|A^{1/2}y|^2)] |A^{1/2}(x - y)|^2.$$

PROOF. Let us set

$$m_x := m(|A^{1/2}x|^2), \quad m_y := m(|A^{1/2}y|^2).$$

Thus an elemental calculation gives:

$$\begin{aligned} \langle m_x Ax - m_y Ay, x - y \rangle &= m_x |A^{1/2}x|^2 + m_y |A^{1/2}y|^2 - (m_x + m_y) \langle Ax, y \rangle \\ &= \frac{1}{2} (m_x + m_y) (|A^{1/2}x|^2 + |A^{1/2}y|^2) - \frac{1}{2} \cdot 2(m_x + m_y) \langle Ax, y \rangle + \\ &\quad + \frac{1}{2} (m_x - m_y) (|A^{1/2}x|^2 - |A^{1/2}y|^2) \\ &= \frac{1}{2} (m_x + m_y) |A^{1/2}(x - y)|^2 + \frac{1}{2} (m_x - m_y) (|A^{1/2}x|^2 - |A^{1/2}y|^2) \\ &\geq \frac{1}{2} (m_x + m_y) |A^{1/2}(x - y)|^2; \end{aligned}$$

where in the last step we exploit that  $m$  is nondecreasing, hence

$$(m(\alpha) - m(\beta))(\alpha - \beta) \geq 0 \quad \forall \alpha, \beta \geq 0.$$

□

The last lemma concerns the integrability properties of the corrector  $\theta_\varepsilon$ .

**Lemma 3.5** *Let  $0 \leq p \leq 1$  and let  $\theta_\varepsilon$  be the solution of (1.6), (1.7). Let  $\delta \geq 0$  and let us assume that  $\varepsilon < (2 + 2\delta)^{-1}$ . Then there exists a constant  $C_\delta$  independent from  $\varepsilon$  and from the initial data such that if  $w_0 \in D(A^{j/2})$  therefore we have*

$$\int_0^{+\infty} (1+t)^\delta |A^{j/2} \theta'_\varepsilon(t)| dt \leq C_\delta |A^{j/2} w_0| \varepsilon.$$

PROOF. Let us define

$$I := \int_0^{+\infty} (1+t)^\delta |A^{j/2} \theta'_\varepsilon(t)| dt.$$

If  $p = 1$  then  $\theta'_\varepsilon(t) = w_0(1+t)^{-1/\varepsilon}$  hence thesis follows from

$$I = |A^{j/2} w_0| \frac{\varepsilon}{1 - (\delta + 1)\varepsilon}.$$

Let us now assume that  $p < 1$ . In such a case we have

$$\theta'_\varepsilon(t) = w_0 \exp \left( -\frac{1}{\varepsilon} \frac{1}{1-p} ((1+t)^{1-p} - 1) \right).$$

If we set

$$\phi(t) := \min\{t, t^{1-p}\}$$

then it is easy to prove that there exists a constant  $\beta_0 > 0$  such that

$$\frac{1}{1-p}((1+t)^{1-p} - 1) \geq \beta_0 \phi(t).$$

In particular we obtain

$$\begin{aligned} I &\leq |A^{j/2}w_0| \int_0^{+\infty} (1+t)^\delta e^{-\frac{1}{\varepsilon}\beta_0\phi(t)} dt \\ &= |A^{j/2}w_0| \left( \int_0^1 (1+t)^\delta e^{-\frac{t}{\varepsilon}\beta_0} dt + \int_1^{+\infty} (1+t)^\delta e^{-\frac{t^{1-p}}{\varepsilon}\beta_0} dt \right). \end{aligned}$$

Let us set  $\varepsilon s = t$ , hence

$$\begin{aligned} I &\leq |A^{j/2}w_0|\varepsilon \left( \int_0^{+\infty} (1+\varepsilon s)^\delta e^{-\beta_0 s} ds + \int_0^{+\infty} (1+\varepsilon s)^\delta e^{-\frac{s^{1-p}}{\varepsilon^p}\beta_0} ds \right) \\ &\leq |A^{j/2}w_0|\varepsilon \left( \int_0^{+\infty} (1+s)^\delta e^{-\beta_0 s} ds + \int_0^{+\infty} (1+s)^\delta e^{-\beta_0 s^{1-p}} ds \right) \\ &= |A^{j/2}w_0|\varepsilon C_\delta. \end{aligned}$$

□

### 3.2 The First order problem

Theory of parabolic equations of Kirchhoff type is quite well established. These equations appeared for the first time in the pioneering paper [2] by S. Bernstein and then were considered by many authors (see [1, 15, 12] and [11] for the details). In fact the following result holds true.

**Theorem 3 (Global solutions)** *Let  $A$  be a nonnegative operator, let  $0 \leq p \leq 1$  and  $\gamma \geq 1$ . Let  $u_0 \in D(A)$ .*

*Then problem (1.5) has a unique global solution*

$$u \in C^1([0, +\infty[; H) \cap C^0([0, +\infty[; D(A)).$$

*If in addition  $A^{1/2}u_0 \neq 0$  then the solution is non-stationary, i.e.  $|A^{1/2}u(t)|^2 \neq 0$  for all  $t \geq 0$  and  $u \in C^\infty([0, +\infty[; D(A^\alpha))$  for every  $\alpha \geq 0$ .*

In the proposition below we collect all the properties of the solutions of (1.5) we need in proof of error estimates. Only some of these properties require  $u_0 \in D(A^{3/2})$ . Nevertheless this is an assumption of Theorem 2.4 hence we do not specify in what cases it is in fact necessary or not.

**Proposition 3.6 (Properties of solutions)** *Let  $u_0 \in D(A^{3/2})$  and let us assume that all conditions of Theorem 3 are verified. Let  $u$  be the global solution of (1.5).*

*Then the following statements hold true.*

- The solution  $u$  verifies the standard estimates below:

$$\frac{|A^{(k+1)/2}u(t)|^2}{|A^{k/2}u(t)|^2} \leq \frac{|A^{(k+1)/2}u_0|^2}{|A^{k/2}u_0|^2}, \quad k = 1, 2, \quad \forall t \geq 0; \quad (3.6)$$

$$\frac{1}{2}|u(t)|^2 + \int_0^t |A^{1/2}u(s)|^{2(\gamma+1)}(1+s)^p ds = \frac{1}{2}|u_0|^2, \quad \forall t \geq 0. \quad (3.7)$$

- The solution  $u$  has these decay properties:

$$\frac{\gamma_3}{(1+t)^{(p+1)/\gamma}} \leq |A^{1/2}u(t)|^2 \leq \frac{\gamma_4}{(1+t)^{(p+1)/(\gamma+1)}}, \quad \forall t \geq 0; \quad (3.8)$$

$$|A^{1/2}u(t)|^{2(\gamma-1)}|Au(t)|^2 \leq \frac{\gamma_4}{(1+t)^{p+1}}, \quad \forall t \geq 0. \quad (3.9)$$

If moreover  $A$  is a coercive operator then

$$|A^{1/2}u(t)|^2 \leq \frac{\gamma_4}{(1+t)^{(p+1)/\gamma}}, \quad \forall t \geq 0. \quad (3.10)$$

- The following integrals are bounded:

$$\int_0^{+\infty} |u'(t)|^2(1+t)^p dt = \int_0^{+\infty} |A^{1/2}u(t)|^{4\gamma}|Au(t)|^2(1+t)^{3p} dt \leq \gamma_5; \quad (3.11)$$

$$\int_0^{+\infty} |A^{1/2}u(t)|^{6\gamma}|A^{3/2}u(t)|^2(1+t)^{5p} dt \leq \gamma_5; \quad (3.12)$$

$$\int_0^{+\infty} [|A^{1/2}u(t)|^{8\gamma}(1+t)^{7p} + |A^{1/2}u(t)|^{6\gamma}(1+t)^{5p}] |A^2u(t)|^2 dt \leq \gamma_5. \quad (3.13)$$

PROOF. From now in most of the proofs we omit the dependence of  $u$  from  $t$ . Moreover often we use that  $0 \leq p \leq 1$  but for shortness sake we do not recall it more. Furthermore we use that in  $]0, +\infty[$  the solution  $u$  is as regular as we want.

*Proof of (3.6)* It is enough to remark that

$$\left( \frac{|A^{(k+1)/2}u|^2}{|A^{k/2}u|^2} \right)' = -2(1+t)^p \frac{|A^{1/2}u|^{2\gamma}}{|A^{k/2}u|^4} (|A^{(k+2)/2}u|^2 |A^{k/2}u|^2 - |A^{(k+1)/2}u|^4) \leq 0,$$

where in the last inequality we exploit that

$$|A^{(k+1)/2}u|^2 = \langle A^{(k+2)/2}u, A^{k/2}u \rangle \leq |A^{(k+2)/2}u| |A^{k/2}u|. \quad (3.14)$$

*Proof of (3.7)* It suffices to integrate in  $[0, t]$  the equality:

$$\left(\frac{1}{2}|u|^2\right)' + |A^{1/2}u|^{2(\gamma+1)}(1+t)^p = 0.$$

*Proof of (3.8)* Using (3.6) with  $k = 1$  we have

$$(|A^{1/2}u|^2)' = -2(1+t)^p|Au|^2|A^{1/2}u|^{2\gamma} \geq -2(1+t)^p \frac{|Au_0|^2}{|A^{1/2}u_0|^2} |A^{1/2}u|^{2(\gamma+1)}.$$

Therefore estimate from below follows from Statement (2) in Lemma 3.2.

Let us now remark that

$$\left((1+t)^{p+1} \frac{|A^{1/2}u|^{2(\gamma+1)}}{2(\gamma+1)}\right)' + (1+t)^{2p+1}|A^{1/2}u|^{4\gamma}|Au|^2 = \frac{p+1}{2(\gamma+1)}(1+t)^p|A^{1/2}u|^{2(\gamma+1)}.$$

Integrating in  $[0, t]$  and using (3.7) we get:

$$(1+t)^{p+1} \frac{|A^{1/2}u(t)|^{2(\gamma+1)}}{2(\gamma+1)} + \int_0^t (1+s)^{2p+1}|A^{1/2}u(s)|^{4\gamma}|Au(s)|^2 ds \leq |u_0|^2 + |A^{1/2}u_0|^{2(\gamma+1)}. \quad (3.15)$$

From this inequality we gain directly the estimate from above in (3.8).

*Proof of (3.9)* Let us define

$$G(t) = (1+t)^{p+1}|A^{1/2}u(t)|^{2(\gamma-1)}|Au(t)|^2.$$

We have to prove that  $G$  is bounded. Taking the time's derivative of  $G$  we obtain

$$G' = \frac{-G}{1+t} \left[ 2|A^{1/2}u|^{2\gamma} \frac{|A^{3/2}u|^2}{|Au|^2} (1+t)^{p+1} + 2(\gamma-1)G - (p+1) \right].$$

Now let us distinguish two cases. If  $\gamma > 1$  we have:

$$G' \leq \frac{-G}{1+t} [2(\gamma-1)G - (p+1)]$$

then thesis follows from Statement (2) in Lemma 3.1.

Instead if  $\gamma = 1$  using (3.14) with  $k = 1$  we get

$$G' \leq \frac{-G}{1+t} [2|Au|^2(1+t)^{p+1} - (p+1)] = \frac{-G}{1+t} [2G - (p+1)],$$

hence we conclude as in the previous case.

*Proof of (3.10)* Since  $\langle Au, u \rangle \geq \nu|u|^2$  then

$$(|A^{1/2}u|^2)' = -2(1+t)^p|Au|^2|A^{1/2}u|^{2\gamma} \leq -2\nu(1+t)^p|A^{1/2}u|^{2(\gamma+1)}.$$

Hence it suffices to apply Statement (1) in Lemma 3.2.

*Proof of (3.11)* Since  $3p \leq 2p+1$  it is a consequence of (3.15).

*Proof of (3.12)* A simple computation gives:

$$\begin{aligned} & \left( \frac{1}{2} |A^{1/2}u|^{4\gamma} |Au|^2 (1+t)^{4p} \right)' + |A^{1/2}u|^{6\gamma} |A^{3/2}u|^2 (1+t)^{5p} = \\ & 2p |A^{1/2}u|^{4\gamma} |Au|^2 (1+t)^{3p} - 2\gamma |A^{1/2}u|^{6\gamma-2} |Au|^4 (1+t)^{5p} \leq 2 |A^{1/2}u|^{4\gamma} |Au|^2 (1+t)^{2p+1}. \end{aligned}$$

Hence thesis follows integrating in  $[0, t]$  and using (3.15).

*Proof of (3.13)* As in the previous case we have

$$\left( \frac{1}{2} |A^{1/2}u|^{6\gamma} |A^{3/2}u|^2 (1+t)^{6p} \right)' + |A^{1/2}u|^{8\gamma} |A^2u|^2 (1+t)^{7p} \leq 3 |A^{1/2}u|^{6\gamma} |A^{3/2}u|^2 (1+t)^{5p}. \quad (3.16)$$

Moreover from (3.6) with  $k=2$  we get

$$\begin{aligned} & \left( \frac{1}{2} |A^{1/2}u|^{4\gamma} |A^{3/2}u|^2 (1+t)^{4p} \right)' + |A^{1/2}u|^{6\gamma} |A^2u|^2 (1+t)^{5p} \leq \\ & 2p |A^{1/2}u|^{4\gamma} |A^{3/2}u|^2 (1+t)^{3p} \leq 2 \frac{|A^{3/2}u_0|^2}{|Au_0|^2} |A^{1/2}u|^{4\gamma} |Au|^2 (1+t)^{3p}. \end{aligned} \quad (3.17)$$

Summing up (3.16) and (3.17), integrating in  $[0, t]$  and using (3.12) and (3.11) we end up with (3.13).

### 3.3 Proof of Theorem 2.1

As in the previous section in most of the proofs we omit the dependence of  $u_\varepsilon$  from  $t$  and we do not recall more that  $0 \leq p \leq 1$ . We divide the proof into three parts. In the first one we state and prove the energy estimates we need, then we prove the existence of global solutions and finally we give the decay estimates.

#### 3.3.1 Basic energy estimates

In this section we prove some estimates that involve the following energies:

$$Q_\varepsilon(t) = \frac{|u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} (1+t)^{1-p}; \quad (3.18)$$

$$D_\varepsilon(t) = \varepsilon \frac{\langle u'_\varepsilon(t), u''_\varepsilon(t) \rangle}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} (1+t)^{2p+1} + \int_0^t \frac{|A^{1/2}u'_\varepsilon(s)|^2}{|A^{1/2}u_\varepsilon(s)|^2} (1+s)^{2p+1} ds; \quad (3.19)$$

$$R_\varepsilon(t) = \left[ \varepsilon \frac{|u''_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} + \frac{|A^{1/2}u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2} \right] (1+t)^{2(p+1)}; \quad (3.20)$$

$$H_\varepsilon(t) = \left[ \varepsilon \frac{|A^{1/2}u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2} + |A^{1/2}u_\varepsilon(t)|^{2(\gamma-1)} |Au_\varepsilon(t)|^2 \right] (1+t)^{p+1}. \quad (3.21)$$

Let us moreover set:

$$h_1 := 4(|u_1|^2 + |A^{1/2}u_0|^{4\gamma} |Au_0|^2) |A^{1/2}u_0|^{-2(\gamma+1)}, \quad h_2 := (\gamma-1)(\sqrt{h_1} + 1) + \sqrt{\gamma-1},$$

$$L_1 := \begin{cases} (3 + 2h_2(\sqrt{h_1} + 1))h_2^2(\gamma-1)^{-2} + H_1(0) + 1 & \text{if } \gamma > 1 \\ 36 + 2|A^{1/2}u_1|^2 |A^{1/2}u_0|^{-2} + 2|Au_0|^2 + 2^{-1} |\langle Au_0, u_1 \rangle| |A^{1/2}u_0|^{-2} & \text{if } \gamma = 1. \end{cases}$$

In the following proposition we recollect all the estimates on (3.18) through (3.21) we need.

**Proposition 3.7 (A priori estimates)** *Let us assume that all the hypotheses of Theorem 2.1 are verified. Then there exists  $\varepsilon_0$  with the following property. If  $\varepsilon \in ]0, \varepsilon_0]$ ,  $S > 0$  and*

$$u_\varepsilon \in C^2([0, S[; H) \cap C^1([0, S[; D(A^{1/2})) \cap C^0([0, S[; D(A))$$

*is a solution of (1.1), (1.2) such that*

$$|A^{1/2}u_\varepsilon(t)|^2 > 0 \quad \forall t \in [0, S[, \quad (3.22)$$

$$\frac{|\langle u'_\varepsilon(t), Au_\varepsilon(t) \rangle|}{|A^{1/2}u_\varepsilon(t)|^2} \leq \frac{K_0}{1+t}, \quad |A^{1/2}u_\varepsilon(t)|^{2(\gamma-1)} |Au_\varepsilon(t)|^2 \leq \frac{K_1}{(1+t)^{p+1}}, \quad \forall t \in [0, S[, \quad (3.23)$$

*then there exists a positive constant  $L_3$  independent from  $\varepsilon$  and  $S$  such that for every  $t \in [0, S[$ :*

$$Q_\varepsilon(t) \leq \max\{4K_1, Q_1(0)\} =: L_2; \quad (3.24)$$

$$D_\varepsilon(t) \leq D_\varepsilon(0) + 2L_2(3 + 2K_0)(1+t)^{p+1} + \frac{1}{8(K_0 + 1)} \int_0^t \frac{|u''_\varepsilon(s)|^2 (1+s)^{2p+1}}{|A^{1/2}u_\varepsilon(s)|^{2(\gamma+1)}} ds; \quad (3.25)$$

$$\left[ \varepsilon \frac{|u''_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} + \frac{|A^{1/2}u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2} \right] (1+t)^{p+1} = \frac{R_\varepsilon(t)}{(1+t)^{p+1}} \leq L_3 + 2R_\varepsilon(0); \quad (3.26)$$

$$H_\varepsilon(t) \leq L_1. \quad (3.27)$$

PROOF. Let us set:

$$h_3 := 4(4\gamma^2 K_0^2 K_1 + L_2), \quad h_4 := h_3 + 8(K_0 + 1)(3 + 2K_0)L_2, \quad (3.28)$$



$$L_3 := 2 \left[ h_4 + \frac{L_2}{2} + 4 \frac{|u_1|}{|A^{1/2}u_0|^{2(\gamma+1)}} (|u_1| + |A^{1/2}u_0|^{2\gamma}|Au_0|)(K_0 + 1) \right]. \quad (3.29)$$

Now let us assume that  $\varepsilon_0$  verifies the following inequalities:

$$8\varepsilon_0(2 + (\gamma + 1)K_0) \leq 1, \quad 16\varepsilon_0(K_0 + 1)^2 \leq 1, \quad (3.30)$$

$$2\varepsilon_0(K_0 + 1)(1 + (3 + 2(\gamma + 1)K_0)^2) \leq 1/8, \quad (3.31)$$

$$\sqrt{\varepsilon_0} \left( \sqrt{L_3} + \sqrt{2} \frac{|A^{1/2}u_1|}{|A^{1/2}u_0|} \right) \leq 1. \quad (3.32)$$

Let us now compute the time's derivatives of the energies (3.18) through (3.21). After some computation we find that:

$$Q'_\varepsilon = Q_\varepsilon \left( \frac{(1-p)}{1+t} - 2(\gamma+1) \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} - \frac{2}{\varepsilon} \frac{1}{(1+t)^p} \right) - \frac{2}{\varepsilon} (1+t)^{1-p} \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2}. \quad (3.33)$$

Let us set:

$$\begin{aligned} \varphi_1(t) &:= \varepsilon \frac{|u''_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} (1+t)^{2p+1} + \\ &+ \varepsilon \left[ \frac{\langle u'_\varepsilon(t), u''_\varepsilon(t) \rangle}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} (1+t)^{2p} \left( 2p+1 - 2(\gamma+1) \frac{\langle Au_\varepsilon(t), u'_\varepsilon(t) \rangle}{|A^{1/2}u_\varepsilon(t)|^2} (1+t) \right) \right], \\ \varphi_2(t) &:= -2\gamma \left( \frac{\langle Au_\varepsilon(t), u'_\varepsilon(t) \rangle}{|A^{1/2}u_\varepsilon(t)|^2} \right)^2 (1+t)^{2p+1}, \\ \varphi_3(t) &:= - \frac{\langle u'_\varepsilon(t), u''_\varepsilon(t) \rangle}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} (1+t)^{p+1}, \\ \varphi_4(t) &:= p \frac{|u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} (1+t)^p; \end{aligned}$$

thus

$$D'_\varepsilon = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4. \quad (3.34)$$

Moreover

$$\begin{aligned} R'_\varepsilon &= 2(1+t)^{2(p+1)} \left( -2\gamma \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^4} \langle u''_\varepsilon, Au_\varepsilon \rangle - \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} \frac{1}{(1+t)^p} \right) + \\ &+ 2(1+t)^{2(p+1)} \left( p \frac{\langle u'_\varepsilon, u''_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} \frac{1}{(1+t)^{p+1}} - \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} \frac{|A^{1/2}u'_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2} \right) + \\ &- 2(\gamma+1)\varepsilon(1+t)^{2(p+1)} \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} + 2(p+1) \frac{R_\varepsilon}{1+t}; \end{aligned} \quad (3.35)$$

$$\begin{aligned}
H'_\varepsilon &= \frac{p+1}{1+t} H_\varepsilon - 2(1+t)^{p+1} \frac{|A^{1/2}u'_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2} \left( \frac{1}{(1+t)^p} + \varepsilon \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} \right) + \\
&\quad + 2(\gamma-1)(1+t)^{p+1} \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} |A^{1/2}u_\varepsilon|^{2(\gamma-1)} |Au_\varepsilon|^2.
\end{aligned} \tag{3.36}$$

We are now ready to prove (3.24) through (3.27).

*Proof of (3.24)* Thanks to (3.33) we have

$$Q'_\varepsilon \leq -\frac{1}{\varepsilon} Q_\varepsilon \left( \frac{2}{(1+t)^p} - \frac{\varepsilon(1-p)}{1+t} + 2\varepsilon(\gamma+1) \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} \right) + \frac{2}{\varepsilon} (1+t)^{1-p} \frac{|u'_\varepsilon| |Au_\varepsilon|}{|A^{1/2}u_\varepsilon|^2}.$$

Moreover by (3.23) and (3.30) we get

$$\begin{aligned}
\frac{2}{(1+t)^p} - \frac{\varepsilon(1-p)}{1+t} + 2\varepsilon(\gamma+1) \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} &\geq \frac{1}{(1+t)^p} (2 - \varepsilon - 2\varepsilon(\gamma+1)K_0) \\
&\geq \frac{7}{4} \frac{1}{(1+t)^p} \geq \frac{1}{(1+t)^p}.
\end{aligned}$$

Hence, using (3.23) once again, we obtain

$$\begin{aligned}
Q'_\varepsilon &\leq -\frac{1}{\varepsilon} \frac{1}{(1+t)^p} Q_\varepsilon + \frac{2}{\varepsilon} \frac{1}{(1+t)^p} \sqrt{Q_\varepsilon} |Au_\varepsilon| |A^{1/2}u_\varepsilon|^{\gamma-1} (1+t)^{(p+1)/2} \\
&\leq -\frac{1}{\varepsilon} \frac{1}{(1+t)^p} \sqrt{Q_\varepsilon} (\sqrt{Q_\varepsilon} - 2\sqrt{K_1}).
\end{aligned}$$

Therefore, since  $Q_\varepsilon(0) = Q_1(0)$ , thesis follows from Statement (1) in Lemma 3.1.

*An intermediate estimate* Thanks to (3.24), for all  $\alpha(t) > 0$  it holds true that in  $[0, S[$ :

$$\begin{aligned}
\frac{|\langle u'_\varepsilon(t), u''_\varepsilon(t) \rangle|}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} &\leq \frac{1}{2} \alpha(t) \frac{|u''_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} + \frac{1}{2\alpha(t)} \frac{|u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} \\
&\leq \frac{1}{2} \alpha(t) \frac{|u''_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}} + \frac{1}{2\alpha(t)} \frac{L_2}{(1+t)^{1-p}}.
\end{aligned} \tag{3.37}$$

*Proof of (3.25)* Let us estimate separately the terms in (3.34).

Using (3.23) and (3.37) with  $\alpha(t) = (1+t)(3+2(\gamma+1)K_0)$  we obtain

$$\begin{aligned}
|\varphi_1| &\leq \varepsilon \left[ \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} (1+t)^{2p+1} + \frac{|\langle u'_\varepsilon, u''_\varepsilon \rangle|}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} (3+2(\gamma+1)K_0) (1+t)^{2p} \right] \\
&\leq \varepsilon \left[ \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} (1+t)^{2p+1} \left( 1 + \frac{1}{2} (3+2(\gamma+1)K_0)^2 \right) + \frac{1}{2} \frac{L_2}{(1+t)^{2-3p}} \right].
\end{aligned}$$

Thus from the smallness assumption (3.31) we get

$$|\varphi_1| \leq \frac{1}{16(K_0 + 1)} \frac{|u_\varepsilon''|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} (1+t)^{2p+1} + \frac{1}{2} L_2 (1+t)^p. \quad (3.38)$$

From (3.37) with  $\alpha(t) = (1+t)^p(8(K_0 + 1))^{-1}$  we have

$$|\varphi_3| \leq \frac{1}{16(K_0 + 1)} \frac{|u_\varepsilon''|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} (1+t)^{2p+1} + 4(K_0 + 1)L_2(1+t)^p. \quad (3.39)$$

Moreover from (3.24) we get

$$|\varphi_4| \leq \frac{L_2}{(1+t)^{1-p}} (1+t)^p \leq L_2(1+t)^p. \quad (3.40)$$

Finally replacing (3.38), (3.39), (3.40) in (3.34), since  $\varphi_2 \leq 0$  we obtain

$$D'_\varepsilon \leq \frac{1}{8(K_0 + 1)} \frac{|u_\varepsilon''|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} (1+t)^{2p+1} + \left( \frac{11}{2} + 4K_0 \right) L_2(1+t)^p.$$

Hence (3.25) follows from a simple integration.

*Proof of (3.26)* Firstly let us estimate some of the terms in (3.35).

Thanks to (3.23) we have

$$\begin{aligned} 2\gamma \frac{|\langle Au_\varepsilon, u'_\varepsilon \rangle|}{|A^{1/2}u_\varepsilon|^2} \frac{|\langle u''_\varepsilon, Au_\varepsilon \rangle|}{|A^{1/2}u_\varepsilon|^2} &\leq 2\gamma \frac{K_0}{1+t} \frac{|u''_\varepsilon|}{|A^{1/2}u_\varepsilon|^{(\gamma+1)}} |Au_\varepsilon| |A^{1/2}u_\varepsilon|^{\gamma-1} \\ &\leq \frac{1}{8} \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} \frac{1}{(1+t)^p} + \frac{8\gamma^2 K_0^2}{(1+t)^{2-p}} |Au_\varepsilon|^2 |A^{1/2}u_\varepsilon|^{2(\gamma-1)} \\ &\leq \frac{1}{8} \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} \frac{1}{(1+t)^p} + \frac{8\gamma^2 K_0^2 K_1}{(1+t)^3}. \end{aligned} \quad (3.41)$$

Moreover from (3.37) with  $\alpha(t) = (1+t)/4$  we achieve

$$\frac{p}{(1+t)^{p+1}} \frac{|\langle u'_\varepsilon, u''_\varepsilon \rangle|}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} \leq \frac{1}{8} \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} \frac{1}{(1+t)^p} + \frac{2L_2}{(1+t)^3}. \quad (3.42)$$

Using once again (3.23) we have

$$\frac{|\langle Au_\varepsilon, u'_\varepsilon \rangle|}{|A^{1/2}u_\varepsilon|^2} \frac{|A^{1/2}u'_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2} \leq \frac{K_0}{1+t} \frac{|A^{1/2}u'_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2}. \quad (3.43)$$

Finally from (3.23) and (3.30) we get also

$$\begin{aligned} \varepsilon(\gamma+1) \frac{|\langle Au_\varepsilon, u'_\varepsilon \rangle|}{|A^{1/2}u_\varepsilon|^2} \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} &\leq \varepsilon \frac{(\gamma+1)K_0}{1+t} \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} \\ &\leq \frac{1}{8} \frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} \frac{1}{(1+t)^p}. \end{aligned} \quad (3.44)$$

Replacing (3.41), (3.42), (3.43), (3.44) in (3.35) and using (3.30) and (3.28) we thus obtain

$$\begin{aligned}
R'_\varepsilon &\leq -\frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}}(1+t)^{p+2}\left(\frac{5}{4}-\frac{2\varepsilon(p+1)}{(1+t)^{1-p}}\right)+ \\
&\quad +2(K_0+p+1)\frac{|A^{1/2}u'_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2}(1+t)^{2p+1}+ \\
&\quad +2(8\gamma^2K_0^2K_1+2L_2)\frac{(1+t)^p}{(1+t)^{1-p}} \\
&\leq -\frac{|u''_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}}(1+t)^{p+2}+4(K_0+1)\frac{|A^{1/2}u'_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2}(1+t)^{2p+1}+h_3(1+t)^p.
\end{aligned}$$

Then integrating in  $[0, t]$  and using (3.25), since  $2p+1 \leq p+2$  we find

$$\begin{aligned}
&R_\varepsilon(t) + \int_0^t \frac{|u''_\varepsilon(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2(\gamma+1)}}(1+s)^{p+2}ds \\
&\leq 4(K_0+1) \int_0^t \frac{|A^{1/2}u'_\varepsilon(s)|^2}{|A^{1/2}u_\varepsilon(s)|^2}(1+s)^{2p+1}ds + \frac{h_3}{1+p}(1+t)^{p+1} + R_\varepsilon(0) \\
&\leq -4(K_0+1) \left[ \varepsilon \frac{\langle u'_\varepsilon(t), u''_\varepsilon(t) \rangle}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}}(1+t)^{2p+1} - D_\varepsilon(0) \right] + \\
&\quad + \frac{1}{2} \int_0^t \frac{|u''_\varepsilon(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2(\gamma+1)}}(1+s)^{p+2}ds + \\
&\quad + 8(K_0+1)(3+2K_0)L_2(1+t)^{p+1} + \frac{h_3}{1+p}(1+t)^{p+1} + R_\varepsilon(0).
\end{aligned}$$

From this inequality and (3.28) it follows that:

$$\begin{aligned}
R_\varepsilon(t) + \frac{1}{2} \int_0^t \frac{|u''_\varepsilon(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2(\gamma+1)}}(1+s)^{p+2}ds &\leq R_\varepsilon(0) + 4(K_0+1)|D_\varepsilon(0)| + \\
&\quad + 4(K_0+1)\varepsilon \frac{|\langle u'_\varepsilon(t), u''_\varepsilon(t) \rangle|}{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}}(1+t)^{2p+1} + h_4(1+t)^{p+1}.
\end{aligned} \tag{3.45}$$

Let us now estimate the terms in the right hand side. Let us remark that from (3.37) with  $\alpha(t) = 4\varepsilon(K_0+1)(1+t)$  we get

$$4(K_0+1)\varepsilon \frac{|\langle u'_\varepsilon, u''_\varepsilon \rangle|}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}}(1+t)^{2p+1} \leq 8\varepsilon^2(K_0+1)^2 \frac{|u''_\varepsilon|^2(1+t)^{2p+2}}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}} + \frac{L_2}{2}(1+t)^{3p-1}.$$

Moreover  $3p-1 \leq 2p \leq p+1$ , hence using also (3.30) we deduce

$$4(K_0+1)\varepsilon \frac{|\langle u'_\varepsilon, u''_\varepsilon \rangle|}{|A^{1/2}u_\varepsilon|^{2(\gamma+1)}}(1+t)^{2p+1} \leq \frac{1}{2}R_\varepsilon + \frac{1}{2}L_2(1+t)^{p+1}. \tag{3.46}$$

Furthermore we have

$$\begin{aligned}
|D_\varepsilon(0)| &\leq \varepsilon |u''(0)| \frac{|u_1|}{|A^{1/2}u_0|^{2(\gamma+1)}} = |u_1 + |A^{1/2}u_0|^{2\gamma}Au_0| \frac{|u_1|}{|A^{1/2}u_0|^{2(\gamma+1)}} \\
&\leq (|u_1| + |A^{1/2}u_0|^{2\gamma}|Au_0|) \frac{|u_1|}{|A^{1/2}u_0|^{2(\gamma+1)}}.
\end{aligned} \tag{3.47}$$

Plugging (3.46) and (3.47) in (3.45) and recalling the definition in (3.29) we get

$$\begin{aligned}
&\frac{1}{2}R_\varepsilon(t) + \frac{1}{2} \int_0^t \frac{|u''_\varepsilon(s)|^2}{|A^{1/2}u_\varepsilon(s)|^{2(\gamma+1)}} (1+s)^{p+2} ds \\
&\leq \left( R_\varepsilon(0) + 4(K_0 + 1)|D_\varepsilon(0)| + h_4 + \frac{1}{2}L_2 \right) (1+t)^{p+1} \\
&\leq \left( R_\varepsilon(0) + \frac{1}{2}L_3 \right) (1+t)^{p+1}.
\end{aligned}$$

Therefore inequality (3.26) is proved.

*Proof of (3.27)* If  $\gamma = 1$ , thesis, eventually for smaller values of  $\varepsilon$ , follows from Theorem 2.2 in [10] (in particular it is a consequence of (3.52)). Then from now let us assume that  $\gamma > 1$ . Let us set

$$F_\varepsilon(t) := \varepsilon \frac{|A^{1/2}u'_\varepsilon(t)|^2}{|A^{1/2}u_\varepsilon(t)|^2} (1+t)^{p+1}, \quad G_\varepsilon(t) := |A^{1/2}u_\varepsilon(t)|^{2(\gamma-1)} |Au_\varepsilon(t)|^2 (1+t)^{p+1}$$

so that  $H_\varepsilon = F_\varepsilon + G_\varepsilon$ . Hence exploiting (3.23) in (3.36) we obtain:

$$H'_\varepsilon \leq -\frac{2}{\varepsilon} F_\varepsilon \left( \frac{1}{(1+t)^p} - \varepsilon(K_0 + p + 1) \frac{1}{1+t} \right) + \frac{p+1}{1+t} G_\varepsilon + 2(\gamma-1) \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} G_\varepsilon.$$

Thanks to (3.30) we have  $1 - \varepsilon(K_0 + 2) \geq 1/2$ , therefore we get

$$H'_\varepsilon \leq -\frac{F_\varepsilon}{\varepsilon} \frac{1}{1+t} + \frac{2}{1+t} G_\varepsilon + 2(\gamma-1) \frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} G_\varepsilon. \tag{3.48}$$

Now let us recall that

$$u'_\varepsilon = -(1+t)^p (\varepsilon u''_\varepsilon + |A^{1/2}u_\varepsilon|^{2\gamma} Au_\varepsilon),$$

hence

$$\frac{\langle Au_\varepsilon, u'_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} = -\varepsilon \frac{\langle Au_\varepsilon, u''_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} (1+t)^p - \frac{1}{1+t} G_\varepsilon.$$

Plugging this identity in (3.48), we arrive at

$$H'_\varepsilon \leq -\frac{1}{1+t} \left( \frac{F_\varepsilon}{\varepsilon} - 2G_\varepsilon + 2(\gamma-1)G_\varepsilon^2 + 2\varepsilon(\gamma-1)(1+t)^{p+1} G_\varepsilon \frac{\langle Au_\varepsilon, u''_\varepsilon \rangle}{|A^{1/2}u_\varepsilon|^2} \right). \tag{3.49}$$

Let us estimate the last term in (3.49). Using (3.26) we obtain

$$\begin{aligned}
2\varepsilon(\gamma-1)(1+t)^{p+1} \frac{|Au_\varepsilon||u_\varepsilon''|}{|A^{1/2}u_\varepsilon|^2} &= 2(\gamma-1)\varepsilon \frac{|u_\varepsilon''|}{|A^{1/2}u_\varepsilon|^{\gamma+1}} (1+t)^{(1+p)/2} \sqrt{G_\varepsilon} \\
&\leq 2(\gamma-1)\sqrt{\varepsilon}\sqrt{G_\varepsilon}\sqrt{2R_\varepsilon(0)+L_3} \\
&\leq 2(\gamma-1)\sqrt{G_\varepsilon} \left( \sqrt{2\varepsilon R_\varepsilon(0)} + \sqrt{\varepsilon}\sqrt{L_3} \right). \quad (3.50)
\end{aligned}$$

By the definition of  $h_1$  moreover it follows that:

$$\begin{aligned}
2\varepsilon R_\varepsilon(0) &= 2\varepsilon^2 \frac{|u_\varepsilon''(0)|^2}{|A^{1/2}u_0|^{2(\gamma+1)}} + 2\varepsilon \frac{|A^{1/2}u_1|^2}{|A^{1/2}u_0|^2} \\
&\leq 4(|u_1|^2 + |A^{1/2}u_0|^{4\gamma}|Au_0|^2) \frac{1}{|A^{1/2}u_0|^{2(\gamma+1)}} + 2\varepsilon \frac{|A^{1/2}u_1|^2}{|A^{1/2}u_0|^2} \\
&= h_1 + 2\varepsilon \frac{|A^{1/2}u_1|^2}{|A^{1/2}u_0|^2}. \quad (3.51)
\end{aligned}$$

Using (3.32) and (3.51), from (3.50) we get

$$2\varepsilon(\gamma-1)(1+t)^{p+1} \frac{|\langle Au_\varepsilon, u_\varepsilon'' \rangle|}{|A^{1/2}u_\varepsilon|^2} \leq 2(\gamma-1)\sqrt{G_\varepsilon}(\sqrt{h_1}+1).$$

Plugging this estimate in (3.49), since  $\varepsilon \leq 1$  we finally achieve

$$H'_\varepsilon \leq -\frac{1}{1+t} \left( F_\varepsilon + 2(\gamma-1)G_\varepsilon^2 - 2G_\varepsilon - 2(\gamma-1)G_\varepsilon^{3/2}(\sqrt{h_1}+1) \right).$$

Since  $H_\varepsilon(0) \leq H_1(0)$ , inequality (3.27) follows recalling the definition of  $L_1$  and applying Lemma 3.3 with

$$a = 2(\gamma-1), \quad b = 2, \quad c = 2(\gamma-1)(\sqrt{h_1}+1).$$

□

### 3.3.2 Existence of global solutions

*Local maximal solutions* Problem (1.1), (1.2) admits a unique local-in-time solution, and this solution can be continued to a solution defined in a maximal interval  $[0, T[$ , where either  $T = +\infty$ , or

$$\limsup_{t \rightarrow T^-} (|A^{1/2}u'_\varepsilon(t)|^2 + |Au_\varepsilon(t)|^2) = +\infty, \quad (3.52)$$

or

$$\liminf_{t \rightarrow T^-} |A^{1/2}u_\varepsilon(t)|^2 = 0. \quad (3.53)$$

We omit the proof of this standard result. The interested reader is referred to [5].

*The standard conserved energy* Let us recall that problem (1.1), (1.2) admits a first order conserved energy, that is

$$\varepsilon |u'_\varepsilon(t)|^2 + \frac{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}}{\gamma+1} + 2 \int_0^t \frac{|u'_\varepsilon(s)|^2}{(1+s)^p} ds = \varepsilon |u_1|^2 + \frac{|A^{1/2}u_0|^{2(\gamma+1)}}{\gamma+1}. \quad (3.54)$$

Therefore  $|A^{1/2}u_\varepsilon(t)|^2$  is bounded independently from  $\varepsilon \leq 1$ .

*Global solutions* We want to apply Proposition 3.7. To this end let us set:

$$K_1 := L_1 + 1 + H_1(0), \quad K_0 := \sqrt{\max\{4K_1, Q_1(0)\}K_1} + \frac{|\langle Au_0, u_1 \rangle|}{|A^{1/2}u_0|^2}. \quad (3.55)$$

For such choices of  $K_0$  and  $K_1$  let us define

$$S := \sup\{\tau \in [0, T[ : (3.22), (3.23) \text{ are verified for all } t \in [0, \tau]\}.$$

Firstly let us remark that since  $|A^{1/2}u_0|^2 > 0$ , for our choices of  $K_0$  and  $K_1$  and  $\varepsilon \leq 1$  we have  $S > 0$ . From now furthermore we assume that  $\varepsilon$  verifies the smallness conditions of Proposition 3.7. Thus in  $[0, S[$  Proposition 3.7 holds true.

We want to prove that  $S = T$ .

Let us assume by contradiction that  $S < T$ . Then by the regularity properties of  $u_\varepsilon$  all the estimates in Proposition 3.7 hold true in  $[0, S]$ . Moreover at least one of the following is verified:

$$|A^{1/2}u_\varepsilon(S)|^2 = 0, \quad (3.56)$$

$$\frac{|\langle u'_\varepsilon(S), Au_\varepsilon(S) \rangle|}{|A^{1/2}u_\varepsilon(S)|^2} = \frac{K_0}{1+S}, \quad (3.57)$$

$$|A^{1/2}u_\varepsilon(S)|^{2(\gamma-1)} |Au_\varepsilon(S)|^2 = \frac{K_1}{(1+S)^{p+1}}. \quad (3.58)$$

*Equality (3.56) is false* Let us set  $y(t) := |A^{1/2}u_\varepsilon(t)|^2$ . Hence by (3.23) in  $[0, S[$  we have:

$$\frac{y'(t)}{y(t)} \geq -\frac{2K_0}{1+t}$$

therefore

$$y(t) \geq y(0)e^{-2K_0 \log(1+t)} = \frac{|A^{1/2}u_0|^2}{(1+t)^{2K_0}}; \quad (3.59)$$

in particular  $|A^{1/2}u_\varepsilon(S)|^2 = y(S) > 0$ .

*Equality (3.57) is false* From (3.24) and (3.27), recalling (3.55) and the definition of  $L_2$  in Proposition 3.7 we indeed have

$$\begin{aligned} \frac{|\langle u'_\varepsilon(S), Au_\varepsilon(S) \rangle|}{|A^{1/2}u_\varepsilon(S)|^2} &\leq \frac{|u'_\varepsilon(S)|}{|A^{1/2}u_\varepsilon(S)|^{\gamma+1}} |Au_\varepsilon(S)| |A^{1/2}u_\varepsilon(S)|^{\gamma-1} \\ &\leq \frac{\sqrt{L_2}}{(1+S)^{(1-p)/2}} \frac{\sqrt{L_1}}{(1+S)^{(1+p)/2}} = \frac{\sqrt{L_2 L_1}}{1+S} < \frac{\sqrt{L_2 K_1}}{1+S} \leq \frac{K_0}{1+S}. \end{aligned}$$

*Equality (3.58) is false* This is an immediate consequence of (3.27).

*Conclusion* We have proved that if  $\varepsilon$  is small enough hence  $S = T$  and in  $[0, T[$  Proposition 3.7 holds true. We need only to prove that  $T = +\infty$ . If it is not the case hence (3.52) or (3.53) hold true. Nevertheless (3.53) is excluded by (3.59). Now let us roll out (3.52). From (3.54) we know that  $|A^{1/2}u_\varepsilon|^2$  is bounded from above, hence from (3.27) we deduce that  $|A^{1/2}u'_\varepsilon|^2$  is bounded. Moreover thanks to (3.59)  $|A^{1/2}u_\varepsilon|^2$  is bounded also from below, thus using once again (3.27) we get that  $|Au_\varepsilon|^2$  is bounded. Hence (3.52) is false.

### 3.3.3 Decay estimates

Since now inequalities (3.27) and (3.24) hold true in  $[0, +\infty[$ , we have already proved (2.3) and (2.4). The estimate from below in (2.2) is a consequence of Proposition 3.3 (see (3.21)) in [10], because we have proved that

$$\frac{|\langle u'_\varepsilon(t), Au_\varepsilon(t) \rangle|}{|A^{1/2}u_\varepsilon(t)|^2} \leq \frac{K_0}{1+t} \quad \forall t \geq 0.$$

*Proof of (2.1), (2.5) and estimate from above in (2.2)* Let us recall that we have already supposed that the smallness assumption (3.30) is verified. We work as in [10], Section 3.4, hence we skip the details. Let us set

$$\mathcal{D}_\varepsilon(t) := \varepsilon(1+t)^p \langle u'_\varepsilon(t), u_\varepsilon(t) \rangle + \frac{1}{2} \left( 1 - \frac{\varepsilon p}{(1+t)^{1-p}} \right) |u_\varepsilon(t)|^2.$$

Since

$$\mathcal{D}'_\varepsilon = -(1+t)^p |A^{1/2}u_\varepsilon|^{2(\gamma+1)} + \varepsilon(1+t)^p |u'_\varepsilon|^2 + \varepsilon \frac{p(1-p)}{2} \frac{|u_\varepsilon|^2}{(1+t)^{2-p}},$$

a simple integration gives

$$\begin{aligned} \int_0^t (1+s)^p |A^{1/2}u_\varepsilon(s)|^{2(\gamma+1)} ds &= \mathcal{D}_\varepsilon(0) - \mathcal{D}_\varepsilon(t) + \varepsilon \int_0^t (1+s)^p |u'_\varepsilon(s)|^2 ds \\ &\quad + \varepsilon \frac{p(1-p)}{2} \int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}} ds. \end{aligned}$$



Moreover, since  $1 - 3\varepsilon \geq 1/4$ , it holds true that

$$-\mathcal{D}_\varepsilon(t) \leq \frac{1}{4}\varepsilon(1+t)^{p+1}|u'_\varepsilon(t)|^2 - \frac{1}{8}|u_\varepsilon(t)|^2,$$

hence we get:

$$\begin{aligned} \frac{1}{8}|u_\varepsilon(t)|^2 + \int_0^t (1+s)^p |A^{1/2}u_\varepsilon(s)|^{2(\gamma+1)} ds &\leq |\mathcal{D}_\varepsilon(0)| + \frac{1}{4}\varepsilon(1+t)^{p+1}|u'_\varepsilon(t)|^2 + \\ &+ \varepsilon \int_0^t (1+s)|u'_\varepsilon(s)|^2 ds + \varepsilon \frac{p(1-p)}{2} \int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}} ds. \end{aligned} \quad (3.60)$$

Let us now define

$$\mathcal{E}_\varepsilon(t) := \left( \varepsilon|u'_\varepsilon(t)|^2 + \frac{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}}{\gamma+1} \right) (1+t)^{p+1}.$$

A simple computation gives

$$\mathcal{E}'_\varepsilon = -(1+t) \left( 2 - \frac{\varepsilon(p+1)}{(1+t)^{1-p}} \right) |u'_\varepsilon|^2 + \frac{p+1}{\gamma+1} (1+t)^p |A^{1/2}u_\varepsilon|^{2(\gamma+1)}.$$

Integrating in  $[0, t]$  and using (3.60) we arrive at

$$\begin{aligned} (1+t)^{p+1} \left( 1 - \frac{p+1}{4(\gamma+1)} \right) \varepsilon|u'_\varepsilon(t)|^2 + \frac{|A^{1/2}u_\varepsilon(t)|^{2(\gamma+1)}}{\gamma+1} (1+t)^{p+1} &\leq \\ &\leq \mathcal{E}_\varepsilon(0) - \left( 2 - \varepsilon(p+1) - \varepsilon \frac{p+1}{\gamma+1} \right) \int_0^t (1+s)|u'_\varepsilon(s)|^2 ds + \\ &+ \frac{p+1}{\gamma+1} \left( |\mathcal{D}_\varepsilon(0)| - \frac{1}{8}|u_\varepsilon(t)|^2 + \varepsilon \frac{p(1-p)}{2} \int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}} ds \right). \end{aligned}$$

Since  $2 - 2\varepsilon(1+p) \geq 1$ , then it holds true that

$$\begin{aligned} \frac{1}{2}\mathcal{E}_\varepsilon(t) + \int_0^t (1+s)|u'_\varepsilon(s)|^2 ds + \frac{1}{8} \frac{p+1}{\gamma+1} |u_\varepsilon(t)|^2 &\leq \\ \mathcal{E}_1(0) + 2|\mathcal{D}_\varepsilon(0)| + \varepsilon \frac{p+1}{\gamma+1} \frac{p(1-p)}{2} \int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}} ds. \end{aligned} \quad (3.61)$$

In particular we have

$$|u_\varepsilon(t)|^2 \leq 8 \frac{\gamma+1}{p+1} (\mathcal{E}_1(0) + 2|\mathcal{D}_\varepsilon(0)|) + 4\varepsilon(1-p) \int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}} ds.$$

From the Gronwall's Lemma we hence get

$$|u_\varepsilon(t)|^2 \leq 16 \frac{\gamma+1}{p+1} (\mathcal{E}_1(0) + 2|\mathcal{D}_\varepsilon(0)|),$$

and finally

$$(1-p) \int_0^t \frac{|u_\varepsilon(s)|^2}{(1+s)^{2-p}} ds \leq 16 \frac{\gamma+1}{p+1} (\mathcal{E}_1(0) + 2|\mathcal{D}_\varepsilon(0)|).$$

Now we go back to (3.61) and from the previous inequality we obtain

$$\frac{1}{2} \mathcal{E}_\varepsilon(t) + \int_0^t (1+s) |u'_\varepsilon(s)|^2 ds + \frac{1}{8} \frac{p+1}{\gamma+1} |u_\varepsilon(t)|^2 \leq 9(\mathcal{E}_1(0) + |u_1||u_0| + |u_0|^2).$$

By this inequality all the estimates we look for immediately follow.  $\square$

### 3.4 Proof of Theorem 2.4

Also in this proof in most cases we omit the dependence of  $u$ ,  $u_\varepsilon$ ,  $\rho_\varepsilon$ ,  $r_\varepsilon$  and  $\theta_\varepsilon$  from  $t$ . Moreover from now on we assume that  $\varepsilon$  verifies the smallness assumptions of Theorem 2.1 in such a way that  $u_\varepsilon$  is globally well defined.

Let us recall that  $r_\varepsilon$  and  $\rho_\varepsilon$  verify the following problems:

$$\begin{cases} \varepsilon r''_\varepsilon + |A^{1/2}u|^{2\gamma} A\rho_\varepsilon + \frac{1}{(1+t)^p} r'_\varepsilon = -\varepsilon u'' + (|A^{1/2}u|^{2\gamma} - |A^{1/2}u_\varepsilon|^{2\gamma}) Au_\varepsilon \\ r_\varepsilon(0) = r'_\varepsilon(0) = 0; \end{cases} \quad (3.62)$$

and

$$\begin{cases} \varepsilon \rho''_\varepsilon + (|A^{1/2}u_\varepsilon|^{2\gamma} Au_\varepsilon - |A^{1/2}u|^{2\gamma} Au) + \frac{1}{(1+t)^p} \rho'_\varepsilon = -\varepsilon u'' \\ \rho_\varepsilon(0) = 0, \rho'_\varepsilon(0) = w_0; \end{cases} \quad (3.63)$$

where  $w_0$ ,  $u_\varepsilon$  and  $u$  are defined in (1.7), (1.1) and (1.5) respectively.

#### 3.4.1 Fundamental energies

Below we define the energies we use in the proof of Theorem 2.4.

Let us set

$$\begin{aligned} D_\rho(t) &:= \int_0^t \langle |A^{1/2}u_\varepsilon(s)|^{2\gamma} Au_\varepsilon(s) - |A^{1/2}u(s)|^{2\gamma} Au(s), \rho_\varepsilon(s) \rangle (1+s)^p ds + \\ &\quad + \varepsilon \langle \rho'_\varepsilon(t), \rho_\varepsilon(t) \rangle (1+t)^p + \frac{1}{2} |\rho_\varepsilon(t)|^2 (1 - \varepsilon p (1+t)^{p-1}); \end{aligned} \quad (3.64)$$

$$E_\rho(t) := (\varepsilon |r'_\varepsilon(t)|^2 + |A^{1/2}u(t)|^{2\gamma} |A^{1/2}\rho_\varepsilon(t)|^2) (1+t)^{2p}; \quad (3.65)$$

$$F_\rho(t) := \varepsilon \frac{|r'_\varepsilon(t)|^2}{|A^{1/2}u(t)|^{2\gamma}} + |A^{1/2}\rho_\varepsilon(t)|^2. \quad (3.66)$$

In the proposition below we recollect all the inequalities verified by these energies we need.

**Proposition 3.8 (Basic error estimates)** *Let  $A$  be a non negative operator and let us assume that  $0 \leq p \leq 1$ ,  $\gamma \geq 1$ . Moreover let us suppose that  $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$ . Then there exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon \leq \varepsilon_0$  and for all  $t \geq 0$  we have*

$$\begin{aligned} |\rho_\varepsilon(t)|^2 + \int_0^t (|A^{1/2}u(s)|^{2\gamma} + |A^{1/2}u_\varepsilon(s)|^{2\gamma})|A^{1/2}\rho_\varepsilon(s)|^2(1+s)^p ds &\leq \\ &\leq \gamma_6\varepsilon^2 + 8\varepsilon \int_0^t |\rho'_\varepsilon(s)|^2(1+s)^p ds; \end{aligned} \quad (3.67)$$

$$F_\rho(t) + \int_0^t \frac{|r'_\varepsilon(s)|^2}{|A^{1/2}u(s)|^{2\gamma}} \frac{1}{(1+s)^p} ds \leq \gamma_7\varepsilon^2 + \gamma_8\varepsilon \int_0^t |\rho'_\varepsilon(s)|^2(1+s)^p ds; \quad (3.68)$$

if moreover (2.7) is verified then

$$E_\rho(t) + \int_0^t |r'_\varepsilon(s)|^2(1+s)^p ds \leq \gamma_9\varepsilon; \quad (3.69)$$

where all constants do not depend on  $\varepsilon$  and  $t$  but only on the initial data.

PROOF. To begin with, we compute the time's derivatives of (3.65), (3.66) and (3.64). Using (3.62) it is easy to see that

$$\begin{aligned} E'_\rho &= -|r'_\varepsilon|^2(1+t)^p \left( 2 - \frac{2\varepsilon p}{(1+t)^{1-p}} \right) + 2p(1+t)^{2p-1}|A^{1/2}u|^{2\gamma}|A^{1/2}\rho_\varepsilon|^2 + \\ &\quad + 2(1+t)^{2p}|A^{1/2}u|^{2\gamma}\langle A\rho_\varepsilon, \theta'_\varepsilon \rangle - 2\varepsilon\langle u'', r'_\varepsilon \rangle(1+t)^{2p} + \\ &\quad + 2(1+t)^{2p}(|A^{1/2}u|^{2\gamma} - |A^{1/2}u_\varepsilon|^{2\gamma})\langle Au_\varepsilon, r'_\varepsilon \rangle + \\ &\quad - 2\gamma(1+t)^{3p}|A^{1/2}u|^{4\gamma-2}|Au|^2|A^{1/2}\rho_\varepsilon|^2 \\ &=: S_1 + S_2 + S_3 + S_4 + S_5 + S_6, \end{aligned} \quad (3.70)$$

and

$$\begin{aligned} F'_\rho &= -\frac{|r'_\varepsilon|^2}{|A^{1/2}u|^{2\gamma}} \left( \frac{2}{(1+t)^p} - 2\gamma\varepsilon|A^{1/2}u|^{2(\gamma-1)}|Au|^2(1+t)^p \right) + 2\langle A\rho_\varepsilon, \theta'_\varepsilon \rangle + \\ &\quad + 2\frac{|A^{1/2}u|^{2\gamma} - |A^{1/2}u_\varepsilon|^{2\gamma}}{|A^{1/2}u|^{2\gamma}}\langle Au_\varepsilon, r'_\varepsilon \rangle - 2\varepsilon\langle u'', r'_\varepsilon \rangle \frac{1}{|A^{1/2}u|^{2\gamma}}. \end{aligned} \quad (3.71)$$

Conversely using (3.63) we have

$$D'_\rho = \varepsilon|\rho'_\varepsilon|^2(1+t)^p - \varepsilon\langle u'', \rho_\varepsilon \rangle(1+t)^p + \frac{p}{2}(1-p)\varepsilon(1+t)^{p-2}|\rho_\varepsilon|^2. \quad (3.72)$$

Moreover from now on let us assume that  $\varepsilon_0$  verifies also these assumptions (recall that we have already supposed that  $\varepsilon_0$  satisfies the smallness conditions in Theorem 2.1)

$$(1+p)\varepsilon_0 \leq \frac{1}{4}, \quad 2\gamma\gamma_4\varepsilon_0 \leq \frac{1}{4} \quad (3.73)$$

where  $\gamma_4$  is the constant in (3.9). In the following we denote by  $c_i$  various constants that depend only on the initial data. Moreover let us set

$$\phi_\rho(t) := (|A^{1/2}u|^{2\gamma} + |A^{1/2}u_\varepsilon|^{2\gamma})|A^{1/2}\rho_\varepsilon|^2.$$

*Preliminary estimates* Thanks to Lagrange's Theorem for all  $t \geq 0$  there exists  $\xi_t$  in the interval with end points  $|A^{1/2}u(t)|^2$  and  $|A^{1/2}u_\varepsilon(t)|^2$  such that

$$\begin{aligned} |A^{1/2}u(t)|^{2\gamma} - |A^{1/2}u_\varepsilon(t)|^{2\gamma} &= \gamma \xi_t^{\gamma-1} (|A^{1/2}u(t)|^2 - |A^{1/2}u_\varepsilon(t)|^2) \\ &= -\gamma \xi_t^{\gamma-1} \langle A^{1/2}(u(t) + u_\varepsilon(t)), A^{1/2}\rho_\varepsilon(t) \rangle. \end{aligned}$$

Since it is clear that

$$\xi_t^{\gamma-1} \leq |A^{1/2}u(t)|^{2(\gamma-1)} + |A^{1/2}u_\varepsilon(t)|^{2(\gamma-1)},$$

then

$$\begin{aligned} (|A^{1/2}u|^{2\gamma} - |A^{1/2}u_\varepsilon|^{2\gamma})^2 &\leq \gamma^2 \xi_t^{2(\gamma-1)} |A^{1/2}(u + u_\varepsilon)|^2 |A^{1/2}\rho_\varepsilon|^2 \\ &\leq 2\gamma^2 (|A^{1/2}u|^{2(\gamma-1)} + |A^{1/2}u_\varepsilon|^{2(\gamma-1)})^2 (|A^{1/2}u|^2 + |A^{1/2}u_\varepsilon|^2) |A^{1/2}\rho_\varepsilon|^2 \\ &\leq 6\gamma^2 (|A^{1/2}u|^{2(\gamma-1)} + |A^{1/2}u_\varepsilon|^{2(\gamma-1)}) \phi_\rho. \end{aligned} \quad (3.74)$$

Moreover computing the time's derivative of (1.5) we get:

$$u'' = -p(1+t)^{p-1} |A^{1/2}u|^{2\gamma} Au + (1+t)^{2p} (|A^{1/2}u|^{4\gamma} A^2u + 2\gamma |A^{1/2}u|^{4\gamma-2} |Au|^2 Au), \quad (3.75)$$

thus

$$\begin{aligned} |u''|^2 &\leq 3(1+t)^{2(p-1)} |A^{1/2}u|^{4\gamma} |Au|^2 + 3(1+t)^{4p} |A^{1/2}u|^{8\gamma} |A^2u|^2 \\ &\quad + 12\gamma^2 (1+t)^{4p} |A^{1/2}u|^{8\gamma-4} |Au|^6. \end{aligned} \quad (3.76)$$

From (3.9) we also deduce that

$$|A^{1/2}u|^{4(\gamma-1)} |Au|^4 \leq \frac{\gamma_4^2}{(1+t)^{2p+2}}. \quad (3.77)$$

We can now estimate the last term in (3.76) using (3.77), so finally we get

$$|u''|^2 \leq c_1 (1+t)^{2p-2} |A^{1/2}u|^{4\gamma} |Au|^2 + 3(1+t)^{4p} |A^{1/2}u|^{8\gamma} |A^2u|^2. \quad (3.78)$$

Now we are ready to prove (3.67), (3.68), (3.69).

*Proof of (3.67)* Thanks to Lemma 3.4 with  $m(r) = r^\gamma$  we have

$$\frac{1}{2} \phi_\rho \leq \langle |A^{1/2}u_\varepsilon|^{2\gamma} Au_\varepsilon - |A^{1/2}u|^{2\gamma} Au, \rho_\varepsilon \rangle$$

hence integrating (3.72) in  $[0, t]$  we get

$$\begin{aligned} &\frac{1}{2} \int_0^t \phi_\rho(s) (1+s)^p ds + \frac{1}{2} |\rho_\varepsilon(t)|^2 (1 - \varepsilon p (1+t)^{p-1}) \\ &\leq -\varepsilon \langle \rho'_\varepsilon(t), \rho_\varepsilon(t) \rangle (1+t)^p + \varepsilon \int_0^t |\rho'_\varepsilon(s)|^2 (1+s)^p ds + \\ &-\varepsilon \int_0^t \langle u''(s), \rho_\varepsilon(s) \rangle (1+s)^p ds + \frac{p}{2} (1-p) \varepsilon \int_0^t (1+s)^{p-2} |\rho_\varepsilon(s)|^2 ds \\ &=: \psi_1 + \psi_2 + \psi_3 + \psi_4. \end{aligned} \quad (3.79)$$

Let us now estimate  $\psi_1, \psi_3, \psi_4$ .

From (2.2) and (2.4) we obtain

$$|u'_\varepsilon|^2 \leq \frac{C_2}{(1+t)^{1-p}} \frac{C_2^{\gamma+1}}{(1+t)^{1+p}} = \frac{C_2^{\gamma+2}}{(1+t)^2},$$

and from (3.9), (3.8) we have

$$|u'|^2 = (1+t)^{2p} |A^{1/2}u|^{4\gamma} |Au|^2 \leq \gamma_4 (1+t)^{2p} \frac{|A^{1/2}u|^{2(\gamma+1)}}{(1+t)^{p+1}} \leq \frac{c_2}{(1+t)^2}.$$

Therefore, recalling that  $|\rho'_\varepsilon|^2 \leq 2(|u'_\varepsilon|^2 + |u'|^2)$  we get

$$|\psi_1| \leq \varepsilon^2 |\rho'_\varepsilon|^2 (1+t)^{2p} + \frac{1}{4} |\rho_\varepsilon|^2 \leq c_3 \varepsilon^2 + \frac{1}{4} |\rho_\varepsilon|^2. \quad (3.80)$$

Now let us estimate  $\psi_3$ . From (3.75) we deduce that

$$\begin{aligned} \langle u'', \rho_\varepsilon \rangle = & -\frac{p}{(1+t)^{1-p}} |A^{1/2}u|^{2\gamma} \langle A^{1/2}u, A^{1/2}\rho_\varepsilon \rangle + (1+t)^{2p} |A^{1/2}u|^{4\gamma} \langle A^{3/2}u, A^{1/2}\rho_\varepsilon \rangle + \\ & + 2\gamma(1+t)^{2p} |A^{1/2}u|^{4\gamma-2} |Au|^2 \langle A^{1/2}u, A^{1/2}\rho_\varepsilon \rangle, \end{aligned}$$

hence

$$\begin{aligned} \varepsilon |\langle u'', \rho_\varepsilon \rangle| (1+t)^p \leq & \frac{1}{4} |A^{1/2}u|^{2\gamma} |A^{1/2}\rho_\varepsilon|^2 (1+t)^p + 3\varepsilon^2 |A^{1/2}u|^{2(\gamma+1)} (1+t)^{3p-2} + \\ & + 3\varepsilon^2 (1+t)^{5p} |A^{1/2}u|^{6\gamma} |A^{3/2}u|^2 + \\ & + 12\gamma^2 \varepsilon^2 (1+t)^{5p} |A^{1/2}u|^{6\gamma-2} |Au|^4. \end{aligned}$$

Using (3.77) to estimate the last term in the previous inequality, since  $3p-2 \leq p$  finally we obtain

$$\begin{aligned} \varepsilon |\langle u'', \rho_\varepsilon \rangle| (1+t)^p \leq & \frac{1}{4} |A^{1/2}u|^{2\gamma} |A^{1/2}\rho_\varepsilon|^2 (1+t)^p + c_4 \varepsilon^2 |A^{1/2}u|^{2(\gamma+1)} (1+t)^p + \\ & + 3\varepsilon^2 (1+t)^{5p} |A^{1/2}u|^{6\gamma} |A^{3/2}u|^2. \end{aligned} \quad (3.81)$$

From (3.81), (3.7), (3.12) thus we arrive at

$$\begin{aligned} |\psi_3| \leq & \frac{1}{4} \int_0^t |A^{1/2}u(s)|^{2\gamma} |A^{1/2}\rho_\varepsilon(s)|^2 (1+s)^p ds + c_5 \varepsilon^2 \\ \leq & \frac{1}{4} \int_0^t \phi_\rho(s) (1+s)^p ds + c_5 \varepsilon^2. \end{aligned} \quad (3.82)$$

Let us now consider  $\psi_4$  and prove that

$$p(1-p) \int_0^t (1+s)^{p-2} |\rho_\varepsilon(s)|^2 ds \leq c_6, \quad \forall t \geq 0. \quad (3.83)$$

If  $p = 1$  thesis is obvious. If  $p < 1$  it is enough to prove that  $|\rho_\varepsilon|^2$  is bounded independently from  $\varepsilon$  and  $t$ . But this is a straightaway consequence of (2.1) and (3.7).

Using (3.83) we then obtain

$$\psi_4 \leq c_6 \varepsilon. \quad (3.84)$$

Now we go back to (3.79), and using (3.80), (3.82), (3.84), since  $p\varepsilon \leq 1/4$  we achieve

$$\frac{1}{4} \int_0^t \phi_\rho(s)(1+s)^p ds + \frac{1}{8} |\rho_\varepsilon(t)|^2 \leq c_7 \varepsilon^2 + c_6 \varepsilon + \varepsilon \int_0^t |\rho'_\varepsilon(s)|^2 (1+s)^p ds. \quad (3.85)$$

Let us now remark that thanks to (2.5) and (3.11) we have

$$\int_0^t |\rho'_\varepsilon(s)|^2 (1+s)^p ds \leq c_8. \quad (3.86)$$

Plugging (3.86) in (3.85) we gain

$$|\rho_\varepsilon|^2 \leq c_9 \varepsilon.$$

At this point we can improve estimates (3.83) and (3.84) as below

$$p(1-p) \int_0^t (1+s)^{p-2} |\rho_\varepsilon(s)|^2 ds \leq c_{10} \varepsilon, \quad \psi_4 \leq c_{10} \varepsilon^2.$$

Using this last estimate in (3.85) instead of (3.84) we finally get

$$\frac{1}{4} \int_0^t \phi_\rho(s)(1+s)^p ds + \frac{1}{8} |\rho_\varepsilon(t)|^2 \leq c_{11} \varepsilon^2 + \varepsilon \int_0^t |\rho'_\varepsilon(s)|^2 (1+s)^p ds.$$

that is (3.67).

*Proof of (3.68)* From (3.9) we have

$$2\gamma |A^{1/2}u|^{2(\gamma-1)} |Au|^2 (1+t)^p \leq \frac{2\gamma\gamma_4}{1+t} \leq \frac{2\gamma\gamma_4}{(1+t)^p},$$

hence from (3.73), (3.71), (3.74) and (3.78) we obtain

$$\begin{aligned} F'_\rho &\leq -\frac{7}{4} \frac{|r'_\varepsilon|^2}{|A^{1/2}u|^{2\gamma}} \frac{1}{(1+t)^p} + 2|A^{1/2}\rho_\varepsilon||A^{1/2}\theta_\varepsilon| + \frac{1}{2} \frac{|r'_\varepsilon|^2}{|A^{1/2}u|^{2\gamma}} \frac{1}{(1+t)^p} + \\ &\quad + 4 \frac{(|A^{1/2}u|^{2\gamma} - |A^{1/2}u_\varepsilon|^{2\gamma})^2}{|A^{1/2}u|^{2\gamma}} |Au_\varepsilon|^2 (1+t)^p + 4\varepsilon^2 \frac{|u''|^2}{|A^{1/2}u|^{2\gamma}} (1+t)^p \\ &\leq -\frac{5}{4} \frac{|r'_\varepsilon|^2}{|A^{1/2}u|^{2\gamma}} \frac{1}{(1+t)^p} + 2|A^{1/2}\rho_\varepsilon||A^{1/2}\theta_\varepsilon| + \\ &\quad + 24\gamma^2 \frac{|Au_\varepsilon|^2 (|A^{1/2}u|^{2(\gamma-1)} + |A^{1/2}u_\varepsilon|^{2(\gamma-1)})}{|A^{1/2}u|^{2\gamma}} \phi_\rho(1+t)^p + \\ &\quad + 4c_1 \varepsilon^2 (1+t)^{3p-2} |A^{1/2}u|^{2\gamma} |Au|^2 + 12\varepsilon^2 (1+t)^{5p} |A^{1/2}u|^{6\gamma} |A^2u|^2. \end{aligned} \quad (3.87)$$

Let us now observe that, thanks to (2.3), (3.8) and (2.2), we have

$$\frac{1}{|A^{1/2}u|^{2\gamma}}|Au_\varepsilon|^2|A^{1/2}u_\varepsilon|^{2(\gamma-1)} + \frac{1}{|A^{1/2}u|^2}|Au_\varepsilon|^2|A^{1/2}u_\varepsilon|^{2(\gamma-1)}\frac{1}{|A^{1/2}u_\varepsilon|^{2(\gamma-1)}} \leq c_{12}.$$

Replacing this inequality in (3.87) and integrating we get

$$\begin{aligned} F_\rho(t) + \frac{5}{4} \int_0^t \frac{|r'_\varepsilon(s)|^2}{|A^{1/2}u(s)|^{2\gamma}} \frac{1}{(1+s)^p} ds &\leq 2 \sup_{0 \leq s \leq t} |A^{1/2}\rho_\varepsilon(s)| \int_0^t |A^{1/2}\theta_\varepsilon(s)| ds + \\ &+ c_{13} \int_0^t (1+s)^p \phi_\rho(s) ds + 4c_1\varepsilon^2 \int_0^t (1+s)^{3p-2} |A^{1/2}u(s)|^{2\gamma} |Au(s)|^2 ds + \\ &+ 12\varepsilon^2 \int_0^t (1+s)^{5p} |A^{1/2}u(s)|^{6\gamma} |A^2u(s)|^2 ds. \end{aligned}$$

We can now use Lemma 3.5 with  $\delta = 0$  and  $j = 1$ , (3.67), (3.6) with  $k = 1$ , (3.13) and (3.7), thus we obtain

$$\begin{aligned} F_\rho(t) + \frac{5}{4} \int_0^t \frac{|r'_\varepsilon(s)|^2}{|A^{1/2}u(s)|^{2\gamma}} \frac{1}{(1+s)^p} ds &\leq c_{14}\varepsilon \sup_{0 \leq s \leq t} |A^{1/2}\rho_\varepsilon(s)| + c_{15}\varepsilon^2 + \\ &+ c_{16}\varepsilon \int_0^t |\rho'_\varepsilon(s)|^2 (1+s)^p ds + 4c_1\varepsilon^2 \frac{|Au_0|^2}{|A^{1/2}u_0|^2} \int_0^t (1+s)^p |A^{1/2}u(s)|^{2(\gamma+1)} ds \\ &\leq \frac{1}{2} \sup_{0 \leq s \leq t} |A^{1/2}\rho_\varepsilon(s)|^2 + c_{17}\varepsilon^2 + c_{16}\varepsilon \int_0^t |\rho'_\varepsilon(s)|^2 (1+s)^p ds. \end{aligned}$$

Let now  $T > 0$  and let us take the essup on  $0 \leq t \leq T$ , then we get

$$\frac{1}{2} \sup_{0 \leq t \leq T} F_\rho(t) + \frac{5}{4} \int_0^T \frac{|r'_\varepsilon(s)|^2}{|A^{1/2}u(s)|^{2\gamma}} \frac{1}{(1+s)^p} ds \leq c_{17}\varepsilon^2 + c_{16}\varepsilon \int_0^T |\rho'_\varepsilon(s)|^2 (1+s)^p ds.$$

Since  $T$  is arbitrary we have proved (3.68).

*Proof of (3.69)* Let us estimate separately the terms  $S_1, \dots, S_5$  in (3.70).

Since  $\varepsilon \leq 1/4$  then

$$S_1 \leq -\frac{3}{2} |r'_\varepsilon|^2 (1+t)^p. \quad (3.88)$$

Moreover

$$S_2 \leq 2(1+t)^p |A^{1/2}u|^{2\gamma} |A^{1/2}\rho_\varepsilon|^2 \leq 2(1+t)^p \phi_\rho. \quad (3.89)$$

Since  $|A^{1/2}u|^{2\gamma}$  is bounded then

$$S_3 \leq c_{18}(1+t)^{2p} |A^{1/2}\rho_\varepsilon| |A^{1/2}\theta'_\varepsilon|. \quad (3.90)$$

From (3.78) we deduce

$$\begin{aligned}
S_4 &\leq \frac{1}{4}|r'_\varepsilon|^2(1+t)^p + 4\varepsilon^2|u''|^2(1+t)^{3p} \\
&\leq \frac{1}{4}|r'_\varepsilon|^2(1+t)^p + 4c_1\varepsilon^2(1+t)^{3p}|A^{1/2}u|^{4\gamma}|Au|^2 + \\
&\quad + 12\varepsilon^2(1+t)^{7p}|A^{1/2}u|^{8\gamma}|A^2u|^2.
\end{aligned} \tag{3.91}$$

Let us now estimate  $S_5$ . From (3.74) and (2.3) we get

$$\begin{aligned}
S_5 &\leq \frac{1}{4}|r'_\varepsilon|^2(1+t)^p + 4(1+t)^{3p}|Au_\varepsilon|^2(|A^{1/2}u|^{2\gamma} - |A^{1/2}u_\varepsilon|^{2\gamma})^2 \\
&\leq \frac{1}{4}|r'_\varepsilon|^2(1+t)^p + c_{19}(1+t)^{2p}|Au_\varepsilon|^2|A^{1/2}u_\varepsilon|^{2(\gamma-1)} \left(1 + \frac{|A^{1/2}u|^{2(\gamma-1)}}{|A^{1/2}u_\varepsilon|^{2(\gamma-1)}}\right) \phi_\rho(1+t)^p \\
&\leq \frac{1}{4}|r'_\varepsilon|^2(1+t)^p + c_{20} \frac{1}{(1+t)^{1-p}} \left(1 + \frac{|A^{1/2}u|^{2(\gamma-1)}}{|A^{1/2}u_\varepsilon|^{2(\gamma-1)}}\right) \phi_\rho(1+t)^p.
\end{aligned} \tag{3.92}$$

Now we want to prove that

$$\chi := \frac{1}{(1+t)^{1-p}} \frac{|A^{1/2}u|^{2(\gamma-1)}}{|A^{1/2}u_\varepsilon|^{2(\gamma-1)}} \leq c_{21}. \tag{3.93}$$

When  $A$  is coercive this is a consequence of (2.2) and (3.10). On the other hand if  $p \leq (\gamma^2 + 1)/(\gamma^2 + 2\gamma - 1)$  then

$$\alpha := \frac{\gamma^2 + 1 - p(\gamma^2 + 2\gamma - 1)}{\gamma(\gamma + 1)} \geq 0,$$

hence from (2.2) and (3.8) we deduce:

$$\chi \leq \frac{c_{22}}{(1+t)^{1-p}} \frac{(1+t)^{(p+1)(\gamma-1)/\gamma}}{(1+t)^{(p+1)(\gamma-1)/(\gamma+1)}} = \frac{c_{22}}{(1+t)^\alpha} \leq c_{22}.$$

At this point from (3.92) and (3.93) it follows that:

$$S_5 \leq \frac{1}{4}|r'_\varepsilon|^2(1+t)^p + c_{23}\phi_\rho(1+t)^p. \tag{3.94}$$

If we put (3.88), (3.89), (3.90), (3.91), (3.94) in (3.70), since  $S_6 \leq 0$  we get:

$$\begin{aligned}
E'_\rho + |r'_\varepsilon|^2(1+t)^p &\leq c_{24}\phi_\rho(1+t)^p + c_{18}(1+t)^{2p}|A^{1/2}\rho_\varepsilon||A^{1/2}\theta'_\varepsilon| + \\
&\quad + 4c_1\varepsilon^2(1+t)^{3p}|A^{1/2}u|^{4\gamma}|Au|^2 + 12\varepsilon^2(1+t)^{7p}|A^{1/2}u|^{8\gamma}|A^2u|^2.
\end{aligned}$$

Integrating in  $[0, t]$  and using (3.67), (3.11), (3.13) we then achieve:

$$\begin{aligned}
E_\rho(t) + \int_0^t |r'_\varepsilon(s)|^2(1+s)^p ds &\leq c_{25}\varepsilon^2 + c_{26}\varepsilon \int_0^t |\rho'_\varepsilon(s)|^2(1+s)^p ds + \\
&\quad + c_{18} \sup_{0 \leq s \leq t} (1+s)^p |A^{1/2}\rho_\varepsilon(s)||A^{1/2}u(s)|^\gamma \int_0^t \frac{|A^{1/2}\theta'_\varepsilon(s)|}{|A^{1/2}u(s)|^\gamma} (1+s)^p ds.
\end{aligned} \tag{3.95}$$



Thanks to (3.8) and Lemma 3.5, with  $j = 1$  and  $\delta = (3p + 1)/2$  (note that thanks to (3.73) all hypotheses of Lemma 3.5 are verified) we have

$$\int_0^t \frac{|A^{1/2}\theta'_\varepsilon(s)|}{|A^{1/2}u(s)|^\gamma} (1+s)^p ds \leq c_{27} \int_0^t |A^{1/2}\theta'_\varepsilon(s)| (1+s)^{(3p+1)/2} ds \leq c_{28}\varepsilon.$$

Finally plugging this inequality in (3.95) and using (3.86) we gain

$$\begin{aligned} E_\rho(t) + \int_0^t |r'_\varepsilon(s)|^2 (1+s)^p ds &\leq c_{25}\varepsilon^2 + c_{29}\varepsilon + c_{30}\varepsilon \sup_{0 \leq s \leq t} (1+s)^p |A^{1/2}\rho_\varepsilon(s)| |A^{1/2}u(s)|^\gamma \\ &\leq c_{31}\varepsilon + \frac{1}{2} \sup_{0 \leq s \leq t} E_\rho(s). \end{aligned}$$

Let now  $T > 0$ , if we take the essup for  $0 \leq t \leq T$  we get

$$\frac{1}{2} \sup_{0 \leq t \leq T} E_\rho(t) + \int_0^T |r'_\varepsilon(s)|^2 (1+s)^p ds \leq c_{31}\varepsilon.$$

Since  $T$  is arbitrary we have proved (3.69).  $\square$

### 3.4.2 Conclusion

From (3.86), (3.68), (3.67) we straight obtain (2.6).

Let us now prove (2.8). From (3.69) and Lemma 3.5 with  $j = 0$  and  $\delta = p$  we get

$$\begin{aligned} \int_0^{+\infty} |\rho'_\varepsilon(s)|^2 (1+s)^p ds &\leq 2 \int_0^{+\infty} |r'_\varepsilon(s)|^2 (1+s)^p ds + 2 \int_0^{+\infty} |\theta'_\varepsilon(s)|^2 (1+s)^p ds \\ &\leq 2\gamma_9\varepsilon + 2\varepsilon C_p \sup_{s \geq 0} |\theta'_\varepsilon(s)| \leq C\varepsilon, \end{aligned}$$

where  $C$  depends only on the data, since  $|\theta'_\varepsilon| \leq |w_0|$ . Finally replacing this estimate in (3.67) and (3.68) we obtain inequality (2.8).

## References

- [1] N. W. BAZLEY, R. J. WEINACHT; A class of explicitly resolvable evolution equations. *Math. Methods Appl. Sci.* **7** (1985), no. 4, 426–431.
- [2] S. BERNSTEIN; Sur une classe d'équations fonctionnelles aux dérivées partielles, (Russian, French summary) *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **4**, (1940). 17–26.
- [3] E. H. DE BRITO; The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability. *Applicable Anal.* **13** (1982), no. 3, 219–233.

- [4] M. GHISI; Global solutions for dissipative Kirchhoff strings with non-Lipschitz nonlinear term. *J. Differential Equations* **230** (2006), no. 1, 128–139.
- [5] M. GHISI, M. GOBBINO; Global existence and asymptotic behavior for a mildly degenerate dissipative hyperbolic equation of Kirchhoff type. *Asymptot. Anal.* **40** (2004), no. 1, 25–36.
- [6] M. GHISI, M. GOBBINO; Global-in-time uniform convergence for linear hyperbolic-parabolic singular perturbations. *Acta Math. Sin. (Engl. Ser.)* **22** (2006), no. 4, 1161–1170.
- [7] M. GHISI, M. GOBBINO; Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: time-decay estimates. *J. Differential Equations* **245** (2008), no. 10, 2979–3007.
- [8] M. GHISI, M. GOBBINO; Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: global-in-time error estimates. *Commun. Pure Appl. Anal.* **8** (2009), no. 4, 1313–1332.
- [9] M. GHISI, M. GOBBINO; Hyperbolic-parabolic singular perturbation for nondegenerate Kirchhoff equations with critical weak dissipation. To appear on *Math. Annal.*
- [10] M. GHISI, M. GOBBINO; Mildly degenerate Kirchhoff equations with weak dissipation: global existence and time decay. *J. Differential Equations* **248** (2010), no. 2, 381–402.
- [11] M. GHISI, M. GOBBINO; Hyperbolic-parabolic singular perturbation for Kirchhoff equations with weak dissipation, *Rend. Ist. Mat. Univ. Trieste* **42** (2010) 67–88.
- [12] M. GOBBINO; Quasilinear degenerate parabolic equations of Kirchhoff type. *Math. Methods Appl. Sci.* **22** (1999), no. 5, 375–388.
- [13] H. HASHIMOTO, T. YAMAZAKI; Hyperbolic-parabolic singular perturbation for quasilinear equations of Kirchhoff type. *J. Differential Equations* **237** (2007), no. 2, 491–525.
- [14] J. L. LIONS; *Perturbations singulières dans les problèmes aux limites et en control optimal*, Lecture Notes in Mathematics, Vol. 323. Springer-Verlag, Berlin-New York, 1973.
- [15] P. D. MILETTA; An evolution equation with nonlocal nonlinearities: existence, uniqueness and asymptotic behavior. *Math. Methods Appl. Sci.* **10** (1988), no. 4, 407–425.

- [16] T. MIZUMACHI; Decay properties of solutions to degenerate wave equations with dissipative terms. *Adv. Differential Equations* **2** (1997), no. 4, 573–592.
- [17] T. MIZUMACHI; Time decay of solutions to degenerate Kirchhoff type equation. *Nonlinear Anal.* **33** (1998), no. 3, 235–252.
- [18] M. NAKAO, J. BAE; Global existence and decay to the initial-boundary value problem for the Kirchhoff type quasilinear wave equation with a nonlinear localized dissipation. *Adv. Math. Sci. Appl.* **13** (2003), no. 1, 165–177.
- [19] K. NISHIHARA, Y. YAMADA; On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms. *Funkcial. Ekvac.* **33** (1990), no. 1, 151–159.
- [20] K. ONO; Sharp decay estimates of solutions for mildly degenerate dissipative Kirchhoff equations. *Kyushu J. Math.* **51** (1997), no. 2, 439–451.
- [21] K. ONO; On global existence and asymptotic stability of solutions of mildly degenerate dissipative nonlinear wave equations of Kirchhoff type. *Asymptot. Anal.* **16** (1998), no. 3-4, 299–314.
- [22] K. ONO; Global solvability for degenerate Kirchhoff equations with weak dissipation. *Math. Japon.* **50** (1999), no. 3, 409–413.
- [23] J. WIRTH; Scattering and modified scattering for abstract wave equations with time-dependent dissipation. *Adv. Differential Equations* **12** (2007), no. 10, 1115–1133.
- [24] Y. YAMADA; On some quasilinear wave equations with dissipative terms. *Nagoya Math. J.* **87** (1982), 17–39.
- [25] T. YAMAZAKI; Asymptotic behavior for abstract wave equations with decaying dissipation. *Adv. Differential Equations* **11** (2006), 419–456.
- [26] T. YAMAZAKI; Hyperbolic-parabolic singular perturbation for quasilinear equations of Kirchhoff type with weak dissipation. *Math. Methods Appl. Sci.* **32** (2009), no. 15, 1893–1918.
- [27] T. YAMAZAKI; Hyperbolic-parabolic singular perturbation for quasilinear equations of Kirchhoff type with weak dissipation of critical power. Preprint.